Université de Montpellier
Laboratoire Montpelliérain Alexander Grothendieck

Programme de Langlands p-modulaire et compatibilité locale-globale

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RÉSUMÉ

Le but de ces *Mémoire d'Habilitation* est de présenter les résultats scientifiques que j'ai obtenus depuis ma thèse de doctorat, en mettant l'accent sur leur contribution au programme de Langlands modulo p et à sa compatibilité locale-globale dans la cohomologie des variétés arithmétiques.

ABSTRACT

The goal of these *Mémoire d'Habilitation* is to present the scientific results I have obtained since my Ph. D. thesis, with a special emphasis to their contribution to the mod p local Langlands program and the local-global compatibility in the cohomology of arithmetic manifolds.

MOTS-CLÉS : Formes automorphes, représentations Galoisiennes, théorie de Hodge p-adique, cohomologie de groupes arithmétiques, programme de Langlands p-adique, conjecture de type Serre, théorèmes de rélévement modulaire.

KEYWORDS : Automorphic forms, Galois representation, p-adic Hodge theory, cohomology of arithmetic groups, p-adic local Langlands, Serre weight conjectures, modularity lifting theorems.

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RINGRAZIAMENTI

Alcuni anni sono passati dalla difesa della mia tesi di dottorato. Non cerco qui un'invocazione alle muse : forse ne sarebbero i tempi, e le circostanze, ma mi trovo in perenne battaglia, immerso in una nebbia che pare interminata, che nasconde la natura delle entità supreme della Matematica. Di queste ho solo certezza che si trovino lì, dappertutto intorno, immutabili dall'inizio dei tempi. Ne sento la grandezza e la vita, ma *quando s'appressano o son, tutto è vano – nostro intelletto.*

Alle persone che mi hanno accompagnato fino a questo punto nel cammino, il più difficile, del lavoro di ricerca matematica -desidero esprimervi la mia riconoscenza. Non potrei essere qui a difendere la mia abilitazione senza il vostro sostegno costante, la vostra pazienza, la vostra generosità.

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Quanta la mia incoscenza quando anni fa ho cominciato a muovere i primi passi nell'immensa foresta (convenientemente p-adicizzata) annunciata da Langlands ! E quanto il coraggio ed i polsi (e la pazienza !) di Ariane Mézard nel guidarmi ! Gliene sono sempre riconoscente.

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Durante questo tempo il mio lavoro è stato guidato dalla ricerca di una corrispondenza di Langlands p-adica. Christophe Breuil e Pierre Colmez sono state luci chiarissime che hanno illuminato il mio animo condividendo la loro ispirazione ed il tempo speso finora insieme -a Orsay, Luminy, Mumbai, Pechino, vicino ad una lavagna o giocando a dadi- è stato inestimabile.

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Ioan Badulescu e Cedric Bonnafé sono stati colleghi straordinari durante questi anni a Montpellier. Hanno avuto la pazienza di ascoltarmi nelle mie speculazioni (...vaneggiamenti ?) su un possibile Langlands p-adico in caratteristica positiva, Jacquet-Langlands, *transfer* endoscopico, varietà di Deligne-Lusztig e deformazioni di algebre di Hecke... È un grandissimo piacere e un onore averli nel Jury.

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Il mio ringraziamento più commosso, perenne, supremo, va ai matematici con cui ho speso la maggiore parte del cammino ! È con massimo trasporto che penso a Brandon (e a Nani !), a Bao (e il piccolo Minh !), a Benjamin -ciao Benj !- (ed il tempo speso a Toronto, all'IHES, a Padova, a Parigi, a Venezia !), a Chol (e la meravigliosa Seoul !), a Dan (e le serate a Marsiglia !), a Florian (e qui sarebbe troppo lunga la lista !), a Karol (e l'abuso di cocktail...). A Yongquan (e Lucy !) ad Andrea (e la sua famiglia !) ed Adriano (di fronte ad un succo di arancia e di limone nel freddo di Grenoble). Lavorare insieme è stata una fonte infinita di vita -di entusiasmo, di ricchezza, gioia, meraviglia, amicizia. Abbiamo condiviso l'entusiasmo e le delusioni della scoperta e della ricerca, avete avuto la pazienza di accogliere i miei limiti nella visione delle cose e spiegarmi pazientemente come arrivare lì dove eravate voi -matematici straordinari e mio ideale di talento e naturalezza-guidandomi in un percorso mille volte accidentato per me.

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Contents

1. Motivation: the mod p local Langlands correspondence

Let p, ℓ be rational primes and fix algebraic closures $\overline{\mathbf{Q}}_p$, $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_p$, $\mathbf{Q}_\ell$ respectively. Let $L/\mathbf{Q}_\ell$ be a finite extension. The local Langlands correspondence, proved independently by work of Harris and Taylor [**HT01**] and Henniart [**Hen00**] (cf. also the more recent work of Scholze [**Sch13**]) gives a unique family of bijections

$$\left\{\begin{array}{l}\text{Irreducible and smooth}\\ \text{representations of } \mathbf{GL}_n(L)\\ \text{ with } \overline{\mathbf{Q}}_p\text{-coefficients}\end{array}\right\} \overset{\mathrm{rec}_{n,K}}{\longrightarrow} \left\{\begin{array}{l}\text{Frobenius semisimple}\\ \text{Weil-Deligne representations}\\ \text{of } W_L,\ n\text{-dimensional over } \overline{\mathbf{Q}}_p\end{array}\right\}$$
$$\pi \longleftrightarrow \mathrm{rec}_{n,K}(\pi)$$

which are compatible with L- and ε-factors of pairs, and satisfy natural compatibilities with contragradient, twist by characters, central character and local class field theory.

Work of Grothendieck [**Gro01**] shows that, as long as $\ell \neq p$, the Galois side of the local correspondence can be reformulated in terms of *continuous* representations of $G_L \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/L)$ (at the price of restricting the local correspondence above to the set of *bounded* Frobenius semisimple Weil-Deligne representations), while we can restrict to *integral* irreducible smooth representations of $\mathbf{GL}_n(L)$ on the representation theoretic side (i.e. consider ℓ-adic Banach space representations), by more recent result of Vignéras [**Vig10**]. This can be seen, in a sense which will be made more precise below, as a byproduct of the compatibility between the local correspondence and the cohomology of arithmetic manifolds.

In the 2-dimensional case the global geometric realization of the local correspondence for $\ell \neq p$ was discovered by Carayol ([**Car86**]) and, years later, by Saito ([**Sai09**]) when $\ell = p$. Let F^+ be a totally real field. For expository reasons we assume that $[F^+ : \mathbf{Q}] > 1$ is odd. Let $G = \mathrm{Res}_{F^+/\mathbf{Q}} B^*$, for a quaternion algebra B with center F^+, which is split at exactly one infinite place, and which is split at all finite places. Given a compact open subgroup $K \leq G(\mathbf{A}^\infty_{F^+})$ we consider the Shimura curve $Y(K)$ over F^+ whose complex points are given by

$$Y(K)(\mathbf{C}) \cong G(F^+)\backslash G(\mathbf{A}^\infty_{F^+}) \times (\mathbf{C} \setminus \mathbf{R})/K.$$

Shrinking the level K, the étale cohomology groups $H^1_{\text{ét}}(Y(K)_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_p)$ form an inductive system endowed with a Galois action. The classical local/global compatibility statement has now the following incarnation:

Theorem 1.1 ([**Car86**]). — *We keep the hypotheses above and we moreover fix an isomorphism $\imath_p : \overline{\mathbf{Q}}_p \to \mathbf{C}$. We have a $G_{F^+} \times G(\mathbf{A}^\infty_{F^+})$ equivariant* *isomorphism*

$$\varinjlim_{K} H^1_{\text{ét}}(Y(K)_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_p) \cong \bigoplus_{\Pi} \imath_p^{-1}(\Pi^\infty) \otimes r_\Pi$$

where the limit runs over compact open subgroups K of $G(\mathbf{A}^\infty_{F^+})$, the direct sum runs over parabolic automorphic representations Π of $G(\mathbf{A}_{F^+})$ with prescribed conditions *at infinite places and r_Π is the G_{F^+}-representation attached to Π (and $\imath_p$) as in* [**Car86**].

Moreover for each finite place v of F^+ such that $v \nmid p$ we have an isomorphism

$$\mathrm{WD}_v(r_\Pi|_{G_{F_v^+}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{2,F_v^+}\left(\imath_p^{-1}(\Pi_v)\right).$$

A few comments here. We wrote $\mathrm{WD}_v(r_\Pi|_{G_{F_v^+}})^{\mathrm{F-ss}}$ to denote Frobenius semisimplification of the Weil-Deligne representation which is attached to $r_\Pi|_{G_{F_v^+}}$ by Grothendieck ℓ-adic monodromy theorem if $v \nmid p$. The smooth $\mathbf{GL}_n(F_v^+)$-representation Π_v is the local constituent at v in the decomposition $\Pi^\infty = \bigotimes'_{v \text{ finite}} \Pi_v$. The prescribed condition for the local component of Π at infinite places is completely explicit, requiring the infinitesimal character of the $(\mathfrak{g}_\infty, \mathfrak{k}_\infty)$-module associated to Π_∞ to be trivial.

Describing the behavior of $r_\Pi|_{G_{F_v^+}}$ at places $v|p$ is more recent and requires tools from p-adic Hodge theory. Even though these are completely different in nature from those used by Grothendieck in ([**Gro01**]), a procedure of Fontaine (cf. e.g. [**PR95**], App. C.1.4) lets us define a Weil-Deligne representation $\mathrm{WD}_v(r_\Pi|_{G_{F_v^+}})$ at places above p. We then have the following local/global compatibility at $\ell = p$:

Theorem 1.2 **([Sai09])**. — *We keep the hypotheses and setting of Theorem 1.1 For each finite place v of F^+ such that $v \mid p$ we have an isomorphism*

$$\mathrm{WD}_v(r_\Pi|_{G_{F_v^+}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{2,F_v^+}\left(\imath_p^{-1}(\Pi_v)\right).$$

Since then much progress has been made. Given a CM field $F/\mathbf{Q}$ we can now associate to a cohomological conjugate self-dual algebraic cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ a continuous Galois representation $r_\Pi : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, which moreover satisfies the local/global compatibility principle $\mathrm{WD}_v(r_\Pi|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{n,F_v}\, \imath_p^{-1}(\Pi_v)$ at all finite places v of F. (This requires geometric methods issued from the étale cohomology of Shimura varieties over CM fields, as pioneered by work of Clozel [**Clo91**], in turn reliying on [**Kot92**], and subsequently developed in full generality by [**Shi11**], [**CH13**], [**BLGGT14**], [**Car14**].)

It is at this point that the p-adic local Langlands program comes into play. By Fontaine's theory, many non-isomorphic de Rham representations

$\rho_v : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ can have isomorphic Weyl-Deligne representation, the reason being that $\mathrm{WD}_v(\rho_v)$ is not equivalent to the datum of the weakly admissible filtered (φ, N, G_{F_v})-module $\mathrm{D}_{\mathrm{pst}}(\rho_v)$ (the Hodge filtration has been forgotten in $\mathrm{WD}_v(\rho_v)$). In particular the local component Π_v of the automorphic representation Π is not sufficient to reconstruct the local Galois parameter $r_\Pi|_{F_v}$ at places v above p.

The p-adic local Langlands program aims at giving a representation-theoretical incarnation of p-adic Hodge theory. Even more, given a continuous Galois representation $\rho_v : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ (resp. $\overline{\rho}_v : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$) one would like to define a p-adic Banach space representation $\Pi(\rho_v)$ of $\mathrm{GL}_n(F_v)$ over $\overline{\mathbf{Q}}_p$ (resp. a smooth $\mathrm{GL}_n(F_v)$ representation $\pi(\overline{\rho}_v)$ over $\mathbf{F}_p$) in a way which is compatible with the classical local Langlands correspondence and with cohomological constructions, and such that the isomorphism class of $\Pi(\rho_v)$ (resp. $\pi(\overline{\rho}_v)$) is uniquely determined by the isomorphism class of ρ_v (resp. $\overline{\rho}_v$).

At the present time the p-adic local Langlands correspondence has been deeply understood in the case of $\mathrm{GL}_2(\mathbf{Q}_p)$ ([**Col10**], [**Eme**], [**CDP14**], [**DLB**]... see [**Ber11**] for an overview). One of the main problems in further generalizations of the local correspondence is that there are not yet enough tools to give a satisfactory classification of p-adic Banach space representations of $\mathrm{GL}_n(F_v)$ over $\overline{\mathbf{Q}}_p$. (Progress in the study of locally analytic representations of $\mathrm{GL}_n(F_v)$ over $\overline{\mathbf{Q}}_p$ has been made in a recent series of papers [**Bre15b**], [**BHS17**], [**BHS**]). In the mod p setting we do have a complete classification of smooth $\mathrm{GL}_n(F_v)$ representations over $\mathbf{F}_p$ in terms of supercuspidal representations ([**Her11**], now generalized further in [**AHHV17**]); unfortunately even a conjectural classification of supercuspidal representations beyond $\mathrm{GL}_2(\mathbf{Q}_p)$ seems far out of reach as for now.

In order to contour this difficulty, and since the local correspondence is expected to satisfy a local/global compatibility principle, a fruitful approach consists in studying those $\mathrm{GL}_n(F_v)$ representations which are obtained from the integral cohomology of arithmetic manifolds.

Let us be more precise and focus on the mod-p situation. We again fix a CM field F/F^+ and let $G_{/F^+}$ be a unitary group such that $G \times_{F^+} F \cong \mathrm{GL}_n$ and $G(F_v^+) \cong U(n)(\mathbf{R})$ at all infinite places v of F^+. For simplicity of exposition we assume that p is inert in F^+, and splits completely in F. In particular by letting $p = ww^c$ in F then $F_p^+ \cong F_w$ and $G(F_p^+) \cong \mathrm{GL}_n(F_w)$; we fix such a choice of $w|p$ in what follows. For a compact open subgroup $K^p \leq G(\mathbf{A}_{F^+}^{(\infty,p)})$ we consider the space of algebraic automorphic forms with infinite level at p:
(1.1)

$$S(K^p, \overline{\mathbf{F}}_p) := \left\{ f : G(F^+)\backslash G(\mathbf{A}_{F^+}^{(\infty)})/K^p \to \overline{\mathbf{F}}_p \mid f \text{ is locally constant} \right\}$$

which is a smooth admissible representation of $G(F_p^+) \cong \mathrm{GL}_n(F_w)$ over $\overline{\mathbf{F}}_p$ ([**Eme06**]). Let $\mathcal{P}$ denote the set of finite primes $w' \in F$ such that $w'|_{F^+}$ is split in F and moreover $K_{w'|_{F^+}}$ is the standard maximal compact (recall the isomorphism $G(F_{w'|_{F^+}}) \cong \mathrm{GL}_n(F_{w'})$). The space (1.1) is endowed with the action of the anemic Hecke algebra $\mathbf{T}^{\mathcal{P}}$ generated by the spherical Hecke operators attached to the places $w' \in \mathcal{P}$. In particular the faithful quotient $\overline{\mathbf{T}}^{\mathcal{P}}$ of $\mathbf{T}^{\mathcal{P}}$ acting on $S(K^p, \overline{\mathbf{F}}_p)$ is a semilocal $\overline{\mathbf{F}}_p$-algebra and we have an Hecke equivariant decomposition

$$S(K^p, \overline{\mathbf{F}}_p) = \bigoplus_{\mathfrak{m} \in \mathrm{Spm}(\overline{\mathbf{T}}^{\mathcal{P}})} S(K^p, \overline{\mathbf{F}}_p)_{\mathfrak{m}}.$$

The scenario suggested by the mod p local Langlands correspondence is the following. Assume, up to shrinking $\mathbf{T}^{\mathcal{P}}$, that there exists a maximal ideal $\mathfrak{m} \in \mathrm{Spm}(\mathbf{T}^{\mathcal{P}})$ such that $S(K^p, \overline{\mathbf{F}}_p)_{\mathfrak{m}} \neq 0$ and a continuous Galois representation $\overline{r}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ satisfying

$$\det\left(1 - \overline{r}_{\mathfrak{m}}^{\vee}(\mathrm{Frob}_{w'})X\right) = \sum_{j=0}^{n} (-1)^j (\mathbf{N}_{F/\mathbf{Q}}(w'))^{\binom{j}{2}} T_{w'}^{(j)} X^j \qquad \mod \mathfrak{m}$$

for all $w' \in \mathcal{P}$ (here, $\mathbf{N}_{F/\mathbf{Q}}$ denotes the norm of F over $\mathbf{Q}$). Then one would expect

$$S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}] = \pi(\overline{r}_{\mathfrak{m}}|_{G_{F_w}})^{n_p}$$

for an integer n_p which depends only on $\overline{r}_{\mathfrak{m}}|_{G_{F_{w'}}}$, $w'|_{F^+} \neq p$ (the local Galois parameters outside p). Hence the very first, necessary questions arise:

1. does the isomorphism class of the smooth representation $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$ depends only on the local Galois parameter $\overline{r}_{\mathfrak{m}}|_{G_{F_w}}$?
2. What is the $\mathrm{GL}_n(F_w)$ and the $\mathrm{GL}_n(\mathscr{O}_{F_w})$-submodule structure of $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$?
3. Is there a "canonical" way to determine the local Galois parameter $\overline{r}_{\mathfrak{m}}|_{G_{F_w}}$ from some representation-theoretic invariants of $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$, e.g. the socle filtration with respect to the standard maximal compact subgroup $\mathrm{GL}_n(\mathscr{O}_{F_w})$, and/or the action of U_p-operators together with carefully chosen group algebra operators for $\mathrm{GL}_n(\mathscr{O}_{F_w})$?

Unfortunately the smooth $\mathrm{GL}_n(F_w)$-representation $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$ above is still poorly understood, except for the $\mathrm{GL}_2(\mathbf{Q}_p)$-case. A first crucial step to describe the structure of $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$ has been provided by the generalizations of Serre's modularity conjecture ([**BDJ10**], [**Gee11b**], [**Sch08**], [**Her09**], [**GHS**]). In fact, by means of local/global compatibility, the data of the weights $W^?(\overline{r}_{\mathfrak{m}}|_{G_{F_w}})$ of $\overline{r}_{\mathfrak{m}}$ is equivalent to the specification of the $\mathrm{GL}_n(\mathscr{O}_{F_w})$-socle of $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$. Moreover in the case of $\mathrm{GL}_2(F_w)$ a natural $\mathrm{GL}_2(\mathbf{F}_w)$-representation $D_0(\overline{r}_{\mathfrak{m}}|_{G_{F_w}})$ (where $\mathbf{F}_w$ denotes the residue field of F_w) was constructed by

Breuil and Paskunas ([**BP12**]), which was conjecturally describing the space of $1+\varpi_w M_2(\mathscr{O}_{F_w})$-invariants of $S(K^p, \overline{\mathbf{F}}_p)[\mathfrak{m}]$.

In a similar flavor Breuil suggested a conjecture ([**Bre14**], Conjecture 1.2) describing part of the integral cohomology $S(K^p, \overline{\mathbf{Z}}_p)[\mathfrak{p}]$ in terms of the local parameter $r_{\mathfrak{p}}$. In this setting $\mathfrak{p}$ denotes a minimal prime of $\mathbf{T}^{\mathcal{P}}$ sitting above $\mathfrak{m}$ for which $S(K^p, \overline{\mathbf{Z}}_p)[\mathfrak{p}] \neq 0$ and $r_{\mathfrak{p}}$ denotes the representation attached to the automorphic form $\Pi_{\mathfrak{p}}$ defined by $S(K^p, \overline{\mathbf{Q}}_p)[\mathfrak{p}]$. Let τ denote the K-type associated to the local component at p of $\Pi_{\mathfrak{p}}$, and assume that τ is moreover *tame*. Breuil's lattice conjecture is a description of the K-representation which is cut out by τ inside $S(K^p, \overline{\mathbf{Z}}_p)[\mathfrak{p}]$, and can thus be seen as a description of the "tame K-typical" part of the integral cohomology $S(K^p, \overline{\mathbf{Z}}_p)[\mathfrak{p}]$.

The above conjectures have been proved in the last few years in the case of $\mathrm{GL}_2(F_w)$, providing spectacular progress in our understanding of the integral cohomology of automorphic forms on unitary groups of rank 2. The methods in the proofs have so far been global in nature and some of the techniques involve the use of *patching functors*, functors defined on the category of $\overline{\mathbf{Z}}_p$-valued representations of $\mathrm{GL}_n(\mathscr{O}_{F_w})$ and which produce modules over a formally smooth modification of local Galois deformation rings (these modules are typically maximally Cohen-Macaulay over their support), and whose fiber at the special point reconstructs the space of integral automorphic forms. All this will be made precise in §4.2 below.

These *Mémoires* aim at presenting the contribution of the author's recent work to the mod p local Langlands program, since the conclusion of his Ph.D. We will first address the case of GL_2 in §2 (where we use both purely local methods §2.1, 2.2, and global methods, §2.3) and then consider the situation for higher rank groups in §3 and 4. In particular Section 3 will be mainly devoted to local/global compatibility problems in the case of $\mathrm{GL}_3(\mathbf{Q}_p)$, with a special emphasis to Hecke actions. Section 4 describes new methods to study potentially crystalline deformation rings beyond the Barsotti-Tate or Fontaine-Laffaille situation, with applications to modularity lifting, Serre and Breuil's conjectures. It also contains new methods (both local and global) to study lattices inside certain Deligne-Lusztig representations.

In what follows we fix a finite extension $E/\mathbf{Q}_p$, with ring of integers $\mathscr{O}$, uniformizer ϖ and residue field $\mathbf{F}$. These will be our rings of coefficients. All representations will be smooth and realizerd over $\mathbf{F}$-vector spaces, unless differently specified. Other notations and conventions will be introduced *au fur et à mesure*.

Proposed Papers for the *Habilitation*

[AM14] Ramla Abdellatif and Stefano Morra. Structure interne des représentations modulo p de $SL_2(\mathbb{Q}_p)$. *Manuscripta Math.*, 143(1-2):191–206, 2014

[HLM17] Florian Herzig, Daniel Le, and Stefano Morra. On mod p local-global compatibility for GL_3 in the ordinary case. *Compositio Math.*, 153(11):2215–2286, 2017

[HMS14] Yongquan Hu, Stefano Morra, and Benjamin Schraen. Sur la fidélité de certaines représentations de $\mathrm{GL}_2(F)$ sous une algèbre d'Iwasawa. *Rend. Semin. Mat. Univ. Padova*, 131:49–65, 2014

[LLHLMb] Dan Le, Bao Le Hung, Brandon Levin, and Stefano Morra. Potentially crystalline deformation rings and Serre-type conjectures (Shapes and Shadows). *Inventiones Mathematicae.* to appear

[LLHLMc] Dan Le, Bao Le Hung, Brandon Levin, and Stefano Morra. Serre weights and Breuil's lattice conjecture in dimension three. `https://arxiv.org/abs/1608.06570`. submitted (2016)

[LM16] Brandon Levin and Stefano Morra. Potentially crystalline deformation rings in the ordinary case. *Annales de l'Institut Fourier*, 66(5):1923–1964, 2016

[LMS] D. Le, S. Morra, and B. Schraen. Multiplicity one at full congruence level. `https://arxiv.org/abs/1608.07987`. submitted (2016)

[LMP] D. Le, S. Morra, and C. Park. On local-global compatibility for GL_3 in the non-ordinary case. `http://www.math.univ-montp2.fr/~morra/non-ordinary.pdf`. submitted (2016)

[Mor17b] Stefano Morra. Sur les atomes automorphes de longueur 2 pour $\mathrm{GL}_2(\mathbf{Q}_p)$. *Doc. Math.*, (22):777–823, 2017

[Mor17a] Stefano Morra. Iwasawa modules and p-modular representations of GL2. *Israel J. Math.*, 219(1):1–70, 2017

[MP] S. Morra and C. Park. Serre weights for three dimensional ordinary galois representations. *Journal of the London Mathematical Society.* to appear

2. The GL_2 situation

In this section we illustrate some progress on the mod p-local Langlands correspondence for $\mathrm{GL}_2(L)$, $L/\mathbf{Q}_p$ being a finite unramified extension, obtained in the work [**Mor17b**], [**Mor17a**], [**LMS**].

We start with a natural continuation of some of the questions treated in the author's Ph.D. thesis, namely the study of invariant spaces, for arbitrarily deep congruence subgroups, of the *atomes automorphes* appearing in the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. The methods are purely local, and make crucial use of the presentation of the *atomes automorphes* appearing in C. Breuil in [**Bre03b**].

In the second subsection we adress the problem of extracting torsion F-modules from certain mod p representations of $\mathrm{GL}_2(L)$. Again, the method is purely local, and shows the great number of étale φ-modules which can be extracted from the cokernel of the standard spherical Hecke operator associated to Serre weights for GL_2.

In the third part we adress a problem of local/global compatibility. There are explicit conjectures on the space of $K(1)$ invariants of the smooth $\mathrm{GL}_2(L)$ representations which are expected to appear in a mod p local Langlands correspondence. We consider this problem in §2.3, where we prove that the smooth representations obtained as Hecke-isotypic spaces of the mod p cohmology of Shimura curves do satisfy the conjectural behavior on $K(1)$ invariant expected by Breuil and Paskunas.

2.1. Invariant spaces under ramified congruence subgroups in the $\mathrm{GL}_2(\mathbf{Q}_p)$ correspondence.

— In this section we address the problem of computing the invariant spaces, under deeply ramified congruence subgroups, of the non-irreducible smooth mod p representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ which naturally appear in the mod p local Langlands correspondence. This is the work contained in [**Mor17b**], a natural continuation of part of the author's Ph.D thesis.

Let ω be the mod p cyclotomic character and un_λ the unramified character of $G_{\mathbf{Q}_p}$ sending the geometric Frobenius to $\lambda \in \mathbf{F}^\times$. An *atome galoisien générique et réductible* (for $G_{\mathbf{Q}_p}$, of dimension 2 over $\mathbf{F}$) is a Galois representation of the form $\begin{pmatrix} \delta_1 & * \\ 0 & \delta_2 \end{pmatrix}$ where $\delta_1 \overset{\mathrm{def}}{=} \omega^{r+1}\mathrm{un}_\lambda$, $r \in \{1, \dots, p-3\}$ and $\delta_2 \overset{\mathrm{def}}{=} \mathrm{un}_{\lambda^{-1}}$. Via the inverse Colmez functor this gives a generator $A_{r,\lambda}$ of the 1-dimensional space

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}\delta_1 \otimes \delta_2\omega^{-1}, \mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}\delta_2 \otimes \delta_1\omega^{-1})$$

(and actually the inverse Colmez functor gives us exactly an extension of principal series).

Let $K := \mathrm{GL}_2(\mathbf{Z}_p)$. For $n \geq 1$ let $K(n)$ be the principal congruence subgroup of niveau n (i.e. the kernel of the reduction morphism $\mathrm{GL}_2(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}_p/p^n)$) and let $I(n+1) \leq K(n)$ be the subgroup formed by those elements of $K(n)$ which are upper triangular modulo $K(n+1)$.

In the following Theorem we give a complete description of the space of $K(n)$ and $I(n)$ invariants of $A_{r,\lambda}$ in generic cases:

Theorem 2.1 (**[Mor17b]**). — *Let $1 \leq r \leq p-3$ and $n \geq 0$. The space of $K(n+1)$-invariants of $A_{r,\lambda}$ is described by the exact sequence*

$$\begin{aligned} 0 \to \quad & \left(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_2 \otimes \delta_1\omega^{-1}\right)^{K(n+1)} \to \left(A_{r,\lambda}\right)^{K(n+1)} \to \\ \to \quad & \left(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_1 \otimes \delta_2\omega^{-1}\right)^{K(n+1)} \to \mathrm{Sym}^{r+2}\mathbf{F}^2 \otimes \det{}^{-1} \to 0. \end{aligned}$$

The space of $I(n+1)$-invariants is described by the exact sequence

$$\begin{aligned} 0 \to \ & \left(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_2 \otimes \delta_1\omega^{-1}\right)^{I(n+1)} \to \left(A_{r,\lambda}\right)^{I(n+1)} \to \\ \to \ & \left(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_1 \otimes \delta_2\omega^{-1}\right)^{I(n+1)} \to (\omega^{r+1} \otimes \omega^{-1}) \oplus (\omega^{-1} \otimes \omega^{r+1}) \to 0. \end{aligned}$$

Let us further remark that even for $r = 0$, $p \geq 3$ and $n = 1$ one can check that the K socle of $A_{r,\lambda}$ coincide with the K socle of the principal series $\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_2 \otimes \delta_1\omega^{-1}$ (the latter being the $\mathrm{GL}_2(\mathbf{Q}_p)$ socle of $A_{r,\lambda}$).

Thanks to Emerton's local/global compatibility [**Eme**] Theorem 2.1 lets us describe Hecke isotypical spaces of the mod p cohomology of the modular curve with *arbitrary level* at p. The following Theorem resumes the global application of Theorem 2.1 as well as the work of [**Mor13**] in the supersingular case:

Theorem 2.2 (**[Mor17b]**). — *Let $p \geq 3$, $\overline{r} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$ a continuous, odd and absolutely irreducible Galois representation. Let Σ_0 be the set of prime divisor of the Artin conductor of $\overline{r}$, $\kappa \in \{2, \ldots, p+1\}$ the minimal weight (up to cyclotomic twist) associated to $\overline{r}$ and let $t \geq 1$. Let $K_p \in \{K(t),\ I(t)\}$ and assume $\kappa \in \{3, \ldots, p-1\}$.*

We fix an admissible level $K_{\Sigma_0} \leq \prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbf{Z}_\ell)$ for $\overline{r}$ in the sense of Emerton and we let $\mathfrak{m}$ be the maximal ideal associated to $\overline{r}$ in the anemic spherical Hecke algebra (outside $\Sigma_0 \cup \{p\}$) acting on the $\mathbf{F}$-valued étale cohomology of the modular curve

$$Y\big(K_p K_{\Sigma_0} K^{\Sigma_0}\big),$$

(*where* $K^{\Sigma_0} \overset{def}{=} \prod_{l \notin \Sigma_0 \cup \{p\}} \mathrm{GL}_2(\mathbf{Z}_\ell)$). *Define finally*

$$d \overset{def}{=} \dim_{\mathbf{F}} \Big(\bigotimes_{\ell \in \Sigma_0} \pi(\overline{r}|_{G_{\mathbf{Q}_\ell}}) \Big)^{K_{\Sigma_0}} \tag{2.1}$$

with $\pi(\overline{r}|_{G_{\mathbf{Q}_\ell}})$ *being the smooth representation of* $\mathrm{GL}_2(\mathbf{Q}_\ell)$ *associated to* $\overline{r}|_{G_{\mathbf{Q}_\ell}}$ *via the Emerton-Helm normalization* ([**EH14**]) *of the mod* p *local Langlands correspondence* ([**Vig01**]).

For $p^t N > 4$ *we have*

$$\dim_{\mathbf{F}} \big(H^1_{\acute{e}t}(Y(K_p K_{\Sigma_0} K^{\Sigma_0})_{\overline{\mathbf{Q}}}, \mathbf{F})[\mathfrak{m}] \big) = \begin{cases} 2d\big(2p^{t-1}(p+1)-4\big) & \text{if } K_p = K(t) \text{ and } \overline{r}|_{G_{\mathbf{Q}_p}} \text{ is absolutely irreducible;} \\ 2d\big(2(p+1)p^{t-1}-(\kappa+1)\big) & \text{if } K_p = K(t) \text{ and } \overline{r}|_{G_{\mathbf{Q}_p}} \text{ is absolutely reducible, non-split;} \\ 2d\big(2(2p^{t-1}-1)\big) & \text{if } K_p = I(t). \end{cases}$$

Note that the value of d can be explicitly described thanks to the work of Emerton-Helm ([**EH14**]), Helm [**Hel13**] and Nadimpalli [**Nad**] on the compatibility of the Artin and automorphic conductors modulo ℓ. For instance if for all $\ell \in \Sigma_0$ the semi-simplification $\big(\overline{r}|_{G_{\mathbf{Q}_\ell}}\big)^{ss}$ is not a twist of $1 \oplus \omega$ nor $1 \oplus 1$, then the subgroup

$$K_{1,\Sigma_0}(N) \overset{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbf{Z}_\ell) \middle| \; c \equiv d-1 \equiv 0 \bmod N \right\}$$

is an admissible level for $\overline{r}$ such that $d = 1$.

2.1.1. Techniques used in the proof. — We briefly describe the techniques involved in the proof of Theorem 2.1. Let us write $M \overset{\text{def}}{=} \big(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_2 \otimes \delta_1 \omega^{-1}\big)|_K$. By formal arguments in homological algebra and thanks to [**BP12**], Theorem 20.3 (stating the injectivity of the K-restriction map for generic $\mathrm{GL}_2(\mathbf{Q}_p)$-extensions of principal series) we have a compatible system of non-zero maps

$$\begin{aligned} &\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbf{Q}_p)}(\mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_1 \otimes \delta_2 \omega^{-1}, \mathrm{Ind}_{\mathrm{B}(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \delta_2 \otimes \delta_1 \omega^{-1}) \hookrightarrow \\ &\hookrightarrow \varprojlim_{n \in \mathbf{N}} \mathrm{Ext}^1_I\big(\big(\mathrm{ind}^I_{\mathrm{B}(\mathbf{Z}_p)} \chi_r \mathfrak{a}\big)^{K_{n+1}}, M\big) \to \\ &\to \mathrm{Ext}^1_{K_0(p^{n+1})}\big(\chi_r \mathfrak{a}, M\big) \end{aligned} \tag{2.2}$$

where $K_0(p^{n+1})$ is the inverse image in K of the standard Borel of $\mathrm{GL}_2(\mathbf{Z}/p^{n+1})$ and $\mathfrak{a}$ is the torus character defined via $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto ad^{-1}$. Write $\mathscr{B}_{n+1}$ to denote the image of $A_{r,\lambda}$ in $\mathrm{Ext}^1_{K_0(p^{n+1})}\big(\chi_r\mathfrak{a}, M\big)$ via (2.2).

We now want to describe in detail the cocycle associated to $\mathscr{B}_{n+1}$. The key point which is available for the $\mathrm{GL}_2(\mathbf{Q}_p)$ setting is an explicit presentation of $A_{r,\lambda}$ given in [**Bre03b**], Corollaire 5.1.4.

Let us write $M|_I = M^+ \oplus M^-$ where $M^\pm$ are uniserial I-module (this is immediate by the Bruhat-Iwahori decomposition). Let $A := \mathbf{F}[\![\mathrm{U}^-(p\mathbf{Z}_p)]\!] \cong \mathbf{F}[\![X]\!]$ be the Iwasawa algebra associated to the lower unipotent $\mathrm{U}^-(p\mathbf{Z}_p)$. Then A is a compact $\mathbf{F}[\![I]\!]$ module in a natural way and we have $(M^-)^\vee|_A \cong A$; in particular we have $\big(M^-\big)^{K(n+1)} \cong A/(X^{p^n})$ for all $n \geq 1$.

The most difficult task is to control the $\mathrm{U}^-(p\mathbf{Z}_p)$-action on the class of $\mathscr{B}_{n+1}$ *modulo* M^-. This is made precise in the following proposition, showing that the required action is as simple as it can possibly be:

Proposition 2.3 **([Mor17b])**. — *Let $n \geq 1$ and consider the extension of discrete $\mathbf{F}[\![K_0(p^{n+1})]\!]$-modules:*

$$0 \to M^+ \oplus M^- \to \mathscr{B}_{n+1} \to \chi_r\mathfrak{a} \to 0.$$

There exists an element $\mathfrak{e}_{n+1} \in \mathscr{B}_{n+1}$, lifting a generator of $\chi_r\mathfrak{a}$, which is fixed *by $\mathrm{U}^-(p^{n+1}\mathbf{Z}_p)$ modulo the one dimensional space $\mathrm{soc}\big(M^+\big)$.*

Moreover for all $\begin{pmatrix} 1+pa & b \\ p^{n+1}c & 1+pd \end{pmatrix} \in K_0(p^{n+1})$ we have $\big(g-1\big)\mathfrak{e}_{n+1} \in \mathrm{soc}(M^+) \oplus \big(M^-\big)^{K_{n+1}}$ and more precisely

$$\big(g-1\big)\mathfrak{e}_{n+1} = c_{r,\lambda,n}\big(\overline{b}X^{p-3-r} - (\overline{a}-\overline{d})(r+2)X^{p-2-r}\big)\cdot\big(M^-\big)^{K_{n+1}}$$

modulo $\mathrm{soc}\big(M^+\big) \oplus \big(X^{p-3-r+(p-2)}\big)\cdot\big(M^-\big)^{K_{n+1}}$, where $c_{r,\lambda,n} \in \mathbf{F}^\times$ depends only on $\mathfrak{e}_{n+1}$, and $A_{r,\lambda}$.

The passage from Proposition 2.3 to the main Theorem 2.1 is performed with technical computations using the standard cohomological theory of p-adic analytic groups. For this aim, a key ingredient is the control of the action of the upper unipotent $\mathrm{U}(\mathbf{Z}_p)$ on the Iwasawa algebra A (with the I-module structure induced by $(M^-)^\vee|_A \cong A$). The following Proposition gives a coarse description of such action, modulo a sufficiently high power of the maximal ideal of A:

Proposition 2.4 **([Mor17b])**. — *Let $m \geq 1$ and $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{B}(\mathbf{Z}_p)$. The action of g on A is described by*

$$g \cdot X^m \in a^m d^{r-m} X^m + \big(X^{m+(p-2)}\big).$$

The previous Proposition can be framed in a much broader context. Linear actions of a uniform p-groups on Iwasawa algebras of p-adic Lie groups have been studied in [**Ard12**], and described in terms of differential operators (cf. Corollary 6.6).

2.2. F-torsion modules & $\mathrm{GL}_2(L)$-representations. — In this section we fix a finite unramified extension $L/\mathbf{Q}_p$ and write $\mathscr{O}_L$ for its ring of integers, $\mathbf{F}_q$ for its residue field. Let π be a smooth $\mathrm{GL}_2(L)$-representation and consider the kernel N_1 of the natural morphism induced by the trace $\mathrm{U}(\mathscr{O}_L) \to \mathrm{U}(\mathbf{Z}_p)$ (where U denotes the standard upper unipotent radical of GL_2). Then the discrete module π^{N_1} becomes a torsion $\mathbf{F}[\![\mathbf{Z}_p]\!][F]$-module, where the variable F satisfies the relation $F([1]-1) = ([1]^p-1)F$. The interest in torsion $\mathbf{F}[\![\mathbf{Z}_p]\!][F]$-modules increased after the work of [**Sch15**], [**Bre15a**], [**Zab**]. In particular Galois representations (either for the group G_L or the product $(G_L)^{[L:\mathbf{Q}_p]}$) are attached to smooth $\mathrm{GL}_2(L)$-representation in a functorial way (using the theory of (φ,Γ)-modules).

But even before that we see that π gives rise (by Pontryagin duality) to a pseudocompact module over $\mathbf{F}[\![\mathscr{O}_L]\!][F]$, where the variable F satisfies $FX_i = X_{i-1}^pF$ for a suitable system of generators $X_0\dots,X_{f-1}$ of the $\mathbf{F}$-algebra $\mathbf{F}[\![\mathscr{O}_L]\!]$.

In this section, which builds on [**Mor17a**], we describe the standard pseudocompact $\mathbf{F}[\![\mathscr{O}_L]\!][F]$-module which is attached to the standard generator of the spherical Hecke algebra for a Serre weight of $\mathrm{GL}_{2/\mathbf{F}_q}$. We prove that such module contains many Γ-stable $\mathbf{F}[\![\mathscr{O}_L]\!]$-submodules of finite colength and which have finite height over $\mathbf{F}[\![\mathscr{O}_L]\!][F]$. (The Γ here is isomorphic to $\mathscr{O}_L^\times$ and is the correct object to consider when working with Lubin-Tate (φ,Γ)-modules.) Moreover the quotient of the standard pseudocompact $\mathbf{F}[\![\mathscr{O}_L]\!][F]$ module by any of the submodules above is $\mathbf{F}[\![\mathscr{O}_L]\!]$-torsion.

We now present the results in more detail. Recall that a *supersingular* representation π of $\mathrm{GL}_2(L)$ is (up to twist) an admissible quotient of an explicit *universal representation* $\pi(\sigma,0)$. The latter is defined to be the cokernel of a standard Hecke operator acting on the compact induction $\mathrm{ind}_{\mathrm{GL}_2(\mathscr{O}_L)L^\times}^{\mathrm{GL}_2(L)}\sigma$ where σ is the $\mathrm{GL}_2(\mathscr{O}_L)L^\times$-inflation of a Serre weight for $\mathrm{GL}_2(\mathbf{F}_q)$ (we fix the action of the uniformizer $\varpi_L \in L^\times$ to be trivial).

Let $I \leq K$ denote the standard Iwahori subgroup (with respect to the upper triangular Borel $\mathrm{B}(\mathbf{F}_q)$). From [**Mor13**], Theorems 1.1 and 1.2 we have a $\mathrm{GL}_2(\mathscr{O}_L)$-equivariant decomposition $\pi(\sigma,0)|_{\mathrm{GL}_2(\mathscr{O}_L)} \cong R_{\infty,0}\oplus R_{\infty,1}$ and the smooth representations $R_{\infty,0}$, $R_{\infty,1}$ fit into an exact sequences:

$$0 \to V_\bullet \to \mathrm{ind}_I^{\mathrm{GL}_2(\mathscr{O}_L)}\big(R_{\infty,\bullet}^-\big) \to R_{\infty,\bullet} \to 0$$

where $V_\bullet$, is an explicit subrepresentation of $\mathrm{ind}_{\mathrm{B}(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi_\bullet$ –the smooth character $\chi_\bullet$ depending in a simple way on the highest weight of σ. (The advantage of the exact sequence above is that it is a priori easier to describe the submodule structure of $\mathbf{F}[\![I]\!]$-modules, in our specific situation the $R^-_{\infty,\bullet}$'s, rather than $\mathbf{F}[\![K]\!]$-modules.)

In Theorem 2.5 we describe the pseudocompact $\mathbf{F}[\![I]\!]$-module associated to $R^-_{\infty,0}$, $R^-_{\infty,1}$. Let $\mathfrak{S}^0_\infty$, $\mathfrak{S}^1_\infty$ be the Pontryagin duals of $R^-_{\infty,0}$, $R^-_{\infty,1}$ respectively. They are profinite modules over the Iwasawa algebra $\mathbf{F}[\![I]\!]$ of I. By restriction, they can equivalently be seen as modules for the Iwasawa algebra $A = F[\![\mathrm{U}^-(\varpi_L\mathscr{O}_L)]\!]$ (here U^- is the standard lower unipotent radical) and endowed with continuous actions of the groups

$$\Gamma \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1+\varpi_L\mathscr{O}_L \end{pmatrix}, \qquad \mathrm{U}(\mathscr{O}_L) \overset{\text{def}}{=} \begin{pmatrix} 1 & \mathscr{O}_L \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \quad \mathrm{T}(\mathbf{F}_q)$$

(we consider $\mathrm{T}(\mathbf{F}_q)$ as a subgroup of $\mathrm{GL}_2(\mathscr{O}_L)$ via the Teichmüller lift).

Note that the actions of Γ, $\mathrm{T}(\mathbf{F}_q)$ on $\mathfrak{S}^0_\infty$, $\mathfrak{S}^1_\infty$ are *semilinear* but $\mathrm{U}(\mathscr{O}_L)$ only acts by continuous $\mathbf{F}$-linear endomorphisms.

In the Propostion below we describe explicitly $\mathfrak{S}^\bullet_\infty$ (for $\bullet \in \{0,1\}$), as a projective limit of finite dimensional A-modules endowed with continuous actions of Γ, $\mathrm{T}(\mathbf{F}_q)$, $\mathrm{U}(\mathscr{O}_L)$. In what follows we write $\underline{r} \in \{0,\dots,p-1\}^f$ for the f-tuple which parametrizes the isomorphism class of $\sigma|_{\mathrm{SL}_2(\mathbf{F}_q)}$, and assume moreover that σ is regular.

Theorem 2.5 **([Mor17a])**. — *Let $\bullet \in \{0,1\}$. The $\mathbf{F}[\![I]\!]$-module $\mathfrak{S}^\bullet_\infty$ is obtained as the limit of a projective system of finite length $\mathbf{F}[\![I]\!]$-modules $\{\mathfrak{S}^\bullet_{n+1}\}_{n\in 2\mathbf{N}+1+\bullet}$ where $\mathfrak{S}^0_1 \cong A/\mathfrak{m}_A$, $\mathfrak{S}^1_2 \cong A/\langle X_i^{(r_i+1)}, i\rangle$ and for all $n \in 2\mathbf{N}+1+\bullet$, $n \geq 2$, the transition morphisms $\mathfrak{S}^\bullet_{n+1} \twoheadrightarrow \mathfrak{S}^\bullet_{n-1}$ fit into the following commutative diagram:*

$$\begin{tikzcd}
\mathfrak{S}^\bullet_{n-1} \arrow[r, hook] & A/\langle X_i^{p^{n-2}(r_{i+n-2}+1)}, i\rangle \arrow[d, hook] & 1 \arrow[d, mapsto] \\
& A/\langle X_i^{p^{n-1}(r_{i+n-1}+1)}, i\rangle & \prod_{i=0}^{f-1} X_i^{p^{n-2}(p(r_{i+n-1}+1)-(r_{i+n-2}+1))} \\
\mathfrak{S}^\bullet_{n+1} \arrow[r, hook] \arrow[uu, two heads] & A/\langle X_i^{p^n(r_{i+n}+1)}, i\rangle \arrow[u, two heads, "\mathrm{proj}_{n+1}"] & \\
\ker(\mathrm{proj}_{n+1}) \arrow[u, hook] \arrow[r, equal] & \ker(\mathrm{proj}_{n+1}) \arrow[u, hook] &
\end{tikzcd}$$

where the left vertical complex is exact and proj_{n+1} *denotes the natural projection.*

We make precise the content of Theorem 2.5. The Iwasawa algebra A can be seen, by the Iwahori decomposition, as a $\mathbf{F}[\![I]\!]$-module. (Recall that A is a complete local regular $\mathbf{F}$ algebra endowed with a regular system of parameters $X_0,\dots,X_{f-1}\in A$, which are eigencharacters for the action of $\mathrm{T}(\mathbf{F}_q)$.) All the morphisms in the diagram (2.3) are $\mathbf{F}[\![I]\!]$-equivariant and one proves that the ideals $\big\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\big\rangle_A$ are stable under the actions of Γ, $\mathrm{T}(\mathbf{F}_q)$, $\mathrm{U}(\mathscr{O}_L)$.

Note that Theorem 2.5 suggests a fractal structure on $\mathfrak{S}_\infty^\bullet$: this is certainly true for $\mathfrak{S}_\infty^\bullet$ as an A-module, and one would hope for the action of $\mathrm{U}(\mathscr{O}_L)$ to be "as simple as possible" (cf. Figure 1). In what follows we determine the $\mathrm{U}(\mathscr{O}_L)$-action, modulo sufficiently high power of the maximal ideal of A.

One can describe precisely the monomorphisms appearing in Theorem 2.5:

$$\mathfrak{S}_{n+1}^\bullet\hookrightarrow A/\big\langle X_i^{p^n(r_{i+n}+1)},\, i=0,\dots,f-1\big\rangle,$$

deducing an explicit family of A-generators $\mathscr{G}_{n+1}^\bullet$ for $\mathfrak{S}_{n+1}^\bullet$ hence a set $\mathscr{G}_\infty^\bullet$ of topological A-generators for $\mathfrak{S}_\infty^\bullet$ (which is infinite as soon as $[L:\mathbf{Q}_p]>1$), and we obtain an A-linear continuous morphism with dense image

$$\prod_{e\in\mathscr{G}_\infty^\bullet} A\cdot e\twoheadrightarrow\mathfrak{S}_\infty^\bullet. \tag{2.3}$$

The torsion properties of $\mathfrak{S}_\infty^\bullet$ are given by the following Theorem:

Theorem 2.6 (**[Mor17a]**). — *For $\bullet\in\{0,1\}$ the module $\mathfrak{S}_\infty^\bullet$ is torsion free over A.*

If $x\in\mathfrak{S}_\infty^\bullet\setminus\{0\}$ is in the image of $\bigoplus_{e\in\mathscr{G}_\infty^\bullet}A\cdot e\to\mathfrak{S}_\infty^\bullet$ then the torsion submodule of $\mathfrak{S}_\infty^\bullet/\langle x\rangle_A$ is dense in $\mathfrak{S}_\infty^\bullet/\langle x\rangle_A$.

The ultimate goal of this construction would be to determine a Lubin-Tate (φ,Γ)-module sitting inside $\mathfrak{S}_\infty^\bullet$, which could be related in some reasonable way with the set of weights associated to a modular Galois representation (of weight σ). (In fact A is endowed with a Frobenius endomorphism $F:A\to A$, which is Γ, $\mathrm{T}(\mathbf{F}_q)$-equivariant.)

The following Theorem exhibits an explicit such module:

Theorem 2.7 (**[Mor17a]**). — *There exists an* explicit $\mathbf{F}[\![I]\!]$ *submodule* $\mathfrak{S}_\infty^{\geqq 1}\oplus\mathfrak{S}_\infty^{\geqq 2}$ *of* $\mathfrak{S}_\infty^0\oplus\mathfrak{S}_\infty^1$ *of finite co-length endowed with an F-semilinear, Γ, $\mathrm{T}(\mathbf{F}_q)$-equivariant endomorphism which we denote as F_∞.*

The topological A-linearization of F_∞

$$A\otimes_{F,A}\big(\mathfrak{S}_\infty^{\geqq 1}\oplus\mathfrak{S}_\infty^{\geqq 2}\big)\overset{id\otimes F_\infty}{\longrightarrow}\mathfrak{S}_\infty^{\geqq 1}\oplus\mathfrak{S}_\infty^{\geqq 2}$$

has image of finite co-length.

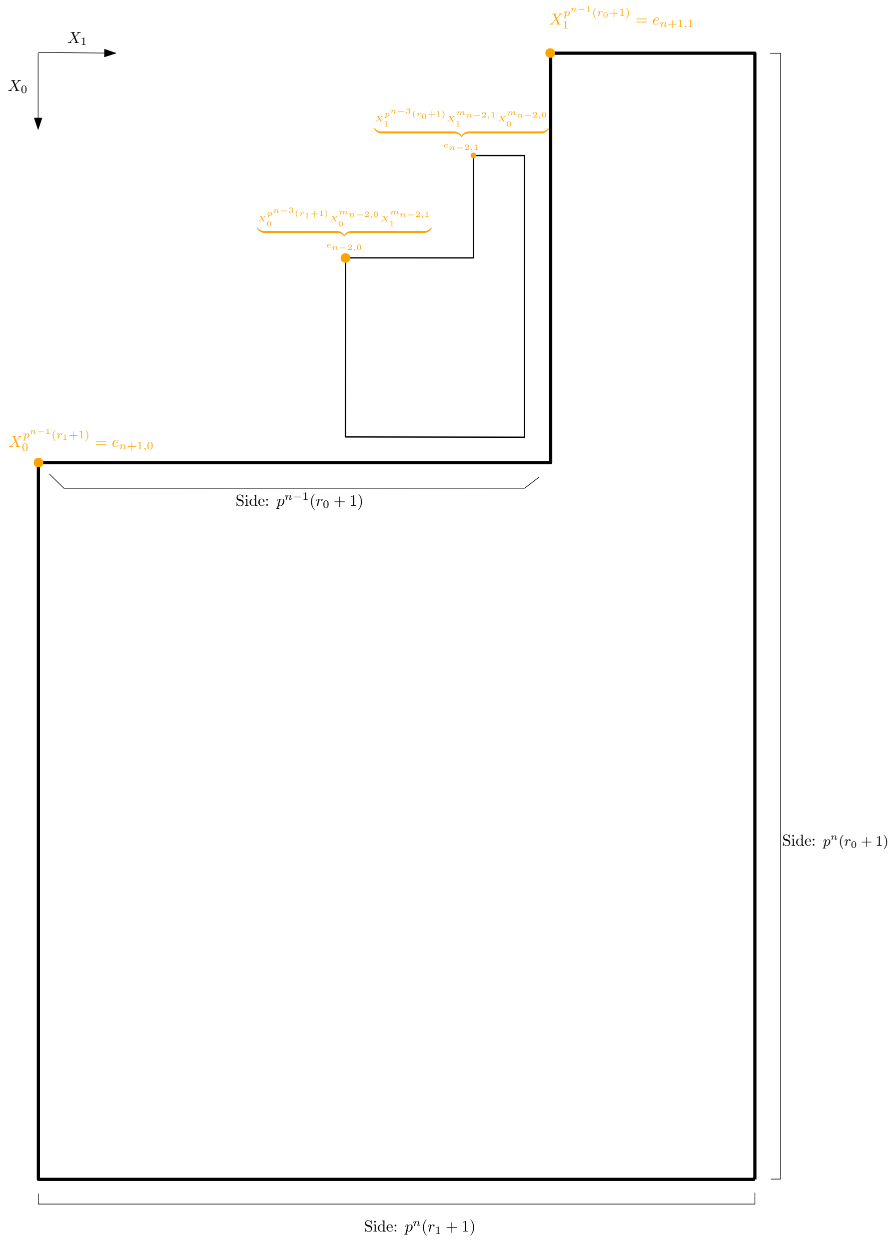

FIGURE 1. The figure represents the fibered product $\ker(\mathrm{proj}_{n+1}) \times_{A_n} \ker(\mathrm{proj}_{n-1})$ over $A_n := A/\langle X_i^{p^{n-1}(r_{i+n-1}+1)}, i\rangle$ appearing in Theorem 2.5, when $f = 2$, $r_0 > r_1$, $n \in 2\mathbf{N}+1$.

Finally, $\mathfrak{S}_\infty^{\geq 1}\oplus\mathfrak{S}_\infty^{\geq 2}$ admits a finite family of generators as a module over the skew power series ring $A[\![F]\!]$, consisting of $[K:\mathbf{Q}_p]$ distinct eigencharacters for the $\mathrm{T}(\mathbf{F}_q)$-action.

We remark that it is possible to find explicit F-stable A-submodules of $\mathfrak{S}_\infty^0\oplus\mathfrak{S}_\infty^1$, of A-rank 2 (hence producing (φ,Γ)-modules over $G_{\mathbf{Q}_p}$ of rank 2). Nevertheless these explicit modules are never Γ-stable, and the resulting $G_{\mathbf{Q}_p}$-representation (in the few explicit cases which have been considered by the author in unpublished computations) does not seem to have a reasonable interpretation in the current mod p local Langlands conjectures.

2.2.1. The case $K=\mathbf{Q}_p$. — For $K=\mathbf{Q}_p$ the Iwasawa module $\mathfrak{S}_\infty^{\geq 1}\oplus\mathfrak{S}_\infty^{\geq 2}$ is free of rank 2 over $A\cong\mathbf{F}[\![X]\!]$. We can describe it in terms of Wach modules.

Let $\omega_2: G_{\mathbf{Q}_{p^2}}\to\mathbf{F}$ be a choice for the Serre fundamental character of niveau 2 (for a fixed embedding $\mathbf{F}_{p^2}\hookrightarrow\mathbf{F}$). For $0\leq r\leq p-1$ we write $\mathrm{ind}(\omega_2^{r+1})$ for the unique (absolutely) irreducible $G_{\mathbf{Q}_p}$-representation whose restriction to the inertia $I_{\mathbf{Q}_p}$ is described by $\omega_2^{r+1}\oplus\omega_2^{p(r+1)}$ and whose determinant is ω^{r+1}. Via the p-modular Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ ([**Bre03a**], Definition 4.2.4), the Galois representation $\mathrm{ind}(\omega_2^{r+1})$ is associated to the supersingular representation $\pi(\sigma_r,0)$.

Then the F_∞-module $\mathfrak{S}_\infty^{\geq 1}\oplus\mathfrak{S}_\infty^{\geq 2}$ associated to $\pi(\sigma_r,0)$ is compatible with the p-modular Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$:

Proposition 2.8 **([Mor17a])**. — *Let $0\leq r\leq p-1$, and write $\chi_{(0,1)}$ for the crystalline character of $G_{\mathbf{Q}_{p^2}}$ such that $\chi_{(0,1)}(p)=1$ and with labelled Hodge-Tate weights $-(0,1)$ (for a choice of an embedding $\mathbf{Q}_{p^2}\hookrightarrow\overline{\mathbf{Q}}_p$). Define the crystalline representation $V_{r+1}\overset{def}{=}\mathrm{ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}}\chi_{(0,1)}^{r+1}$.*

Then we have an isomorphism of F-modules

$$\mathfrak{S}_\infty^{\geq 1}\oplus\mathfrak{S}_\infty^{\geq 2}\xrightarrow{\sim}\mathbf{N}(\overline{V}_{r+1})$$

where $\mathbf{N}(\overline{V}_{r+1})$ is the mod-p reduction of the Wach module associated to V_{r+1} and $\mathfrak{S}_\infty^{\geq 1}\oplus\mathfrak{S}_\infty^{\geq 2}$ is the Iwasawa module of Theorem 2.7.

We recall that the mod-p reduction of V_{r+1} is the dual of $\mathrm{ind}(\omega_2^{r+1})$; in particular the statement of Proposition 2.8 is consistent with the p-modular Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$.

2.2.2. Comparison between A-radical and $\mathbf{F}[\![I]\!]$-radical filtrations. — We conclude with some remarks concerning the radical filtration of $\mathfrak{S}_\infty^\bullet$ *as an $\mathbf{F}[\![I]\!]$-module*. This is a difficult problem: provided the surjection (2.3) none of the composite morphisms $A\hookrightarrow\mathfrak{S}_\infty^\bullet$ are equivariant for the extra actions of Γ, $\mathrm{U}(\mathscr{O}_L)$.

Theorem 2.9 (**[Mor17a]**). — *Let $\bullet \in \{0,1\}$ and, for $k \in \mathbf{N}$, write $\mathscr{I}_k$ for the closure of $\mathfrak{m}_A^k \mathfrak{S}_\infty^\bullet$ in $\mathfrak{S}_\infty^\bullet$ (where $\mathfrak{m}_A$ denotes the maximal ideal of A).*

Then the A-linear filtration $\{\mathscr{I}_k\}_{k\in\mathbf{N}}$ coincides with the $\mathbf{F}[\![I]\!]$-radical filtration on $\mathfrak{S}_\infty^\bullet$.

The main difficulty in obtaining 2.9 consits in a sufficiently precise control of the $\mathbf{F}[\![\mathrm{U}(\mathscr{O}_L)]\!]$-action on the kernel of the transition maps $\ker\big(\mathfrak{S}_{n+1}^\bullet \twoheadrightarrow \mathfrak{S}_{n-1}^\bullet\big)$. This is more subtle than controling the $\mathbf{F}[\![\mathrm{U}(\mathscr{O}_L)]\!]$-action on A (which is doable via [**Ard12**], Corollary 6.6). In our context we use in a crucial way the properties of the Frobenius morphism on the set of the canonical generators of $\ker\big(\mathfrak{S}_{n+1}^\bullet \twoheadrightarrow \mathfrak{S}_{n-1}^\bullet\big)$.

A direct consequence of Theorem 2.9 is the description of the topological generators of $\mathfrak{S}_\infty^\bullet$ (which are dual to the $I(1)$-fixed vectors of $R_\infty^\bullet$). If χ is an I-character, we write $V(\chi)$ to denote the χ-isotypical component of the $\mathbf{F}[\![I]\!]$-cosocle of $\mathfrak{S}_\infty^\bullet$.

Corollary 2.10. — *Assume that σ is a regular Serre weight. Then*

$$\begin{aligned} \mathit{cosoc}_{\mathbf{F}[\![I]\!]}\big(\mathfrak{S}_\infty^0\big) &= V(\chi_{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}}\det{}^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}) \\ \mathit{cosoc}_{\mathbf{F}[\![I]\!]}\big(\mathfrak{S}_\infty^1\big) &= V(\chi_{\underline{r}}\det{}^{-\underline{r}}) \oplus \bigoplus_{i=0}^{f-1} V(\chi_{\underline{r}}\det{}^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)}) \end{aligned}$$

where

$$\dim(V(\chi_{-\underline{r}})) = \dim(V(\chi_{\underline{r}}\det{}^{-\underline{r}})) = 1,$$

$$\dim(V(\chi_{\underline{r}}\det{}^{-\underline{r}}\mathfrak{a}^{-p^i(r_i+1)})) = \begin{cases} \infty & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } L \neq \mathbf{Q}_p \\ 0 & \textit{for all } i \in \{0,\dots,f-1\} \textit{ if } L = \mathbf{Q}_p. \end{cases}$$

Here, $\chi_{\underline{r}}$, $\mathfrak{a}$ are the smooth characters of I characterized by

$$\chi_{\underline{r}}\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \overline{a}^{\sum_{i=0}^{f-1} p^i r_i}, \qquad \mathfrak{a}\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \overline{ad}^{-1}.$$

We hope the techniques involved in Theorem 2.9, combined with the combinatoric of Serre weight for a modular Galois representation, could ultimately lead to the construction of explicit admissible quotients of the standard representation $\pi(\sigma_{\underline{r}}, 0)$ (cf. partial progress in [**Hen**]).

2.3. Multiplicity one at full level: Shimura Curves. — We now move to a problem associated to local/global compatibility in the mod p local Langlands program for Hilbert modular forms.

We fix a totally real field $F^+/\mathbf{Q}$ and a modular Galois representation $\overline{r}: G_{F^+} \to \mathrm{GL}_2(\mathbf{F})$; let $\mathfrak{m}_{\overline{r}}$ be the corresponding Hecke eigensystem and fix a place

$v|p$ of F^+. The mod p local-global compatibility principle predicts that the $\mathfrak{m}_{\overline{r}}$-torsion subspace in the mod p cohomology of a Shimura curve with infinite level at v realizes the mod p Langlands correspondence for $\mathrm{GL}_2(F_v^+)$ (see [**Bre10**]), generalizing the case of modular curves ([**Col10, Eme, Ps13**]). Let π denote the $\mathfrak{m}_{\overline{r}}$-torsion subspace above. The goal of the mod p local Langlands program is then to describe π in terms of the local Galois parameter $\overline{r}|_{G_{F_v^+}}$, though it is not even known whether π depends only on $\overline{r}|_{G_{F_v^+}}$. One of the major difficulties is that little is known about supersingular representations outside of the case of $\mathrm{GL}_2(\mathbf{Q}_p)$ (see [**AHHV17**]).

We now assume that p is unramified in F^+ and that $\overline{r}|_{G_{F_v^+}}$ is 1-generic. Let $K = \mathrm{GL}_2(\mathscr{O}_{F_v^+})$ and $I(1) \subset K$ be the usual pro-p Iwahori subgroup. In [**BDJ10**] and [**Bre14**, Conjecture B.1] we find a conjectural description of the K-socle and $I(1)$-invariants of π—in particular they should satisfy mod p multiplicity one when the tame level is minimal. The work of [**Gee11a**] and [**EGS15**] later confirmed these conjectures. In [**Bre14**], Breuil shows that such a π (also satisfying other properties known for $\mathfrak{m}_{\overline{r}}$-torsion in completed cohomology) must contain a member of a family of representations constructed in [**BP12**]. If $f = 1$, this family has one element, and produces the (one-to-one) mod p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. For $f > 1$, each family is infinite (see [**Hu10**]), and so a naïve one-to-one correspondence cannot exist. Moreover, the K-socle and the $I(1)$-invariants are not sufficient to specify a single mod p representation of $\mathrm{GL}_2(\mathbf{Q}_{p^f})$ when $f > 1$.

However, [**EGS15**] proves a stronger multiplicity one result than what is used in the construction of [**BP12**], namely a result for any lattice in a tame type with irreducible cosocle. We strengthen this result in tame situations as follows. Let $K(1) \subset K$ be the principal congruence subgroup. Assume that in the definition of π we consider the cohomology of a Shimura curve with infinite level at v and minimal tame level.

Theorem 2.11 **([LMS])**. — *Suppose that $\overline{r}$ is 1-generic and tamely ramified at v and satisfies the Taylor–Wiles hypotheses. Let $\mathbf{F}_q$ denote the residue field of F_v^+. Then the $\mathrm{GL}_2(\mathbf{F}_q)$-representation $\pi^{K(1)}$ is isomorphic to the representation $D_0(\overline{r}|_{G_{F_v^+}})$ (which depends only on $\overline{r}|_{I_{F_v^+}}$) constructed in* [**BP12**]. *In particular, its Jordan–Hölder constituents appear with multiplicity one.*

If the Jordan–Hölder constituents of a $\mathrm{GL}_2(\mathbf{F}_q)$-representation appear with multiplicity one, we say that the representation is *multiplicity free*. In particular for $p > 3$, there exists a supersingular $\mathrm{GL}_2(F_v^+)$-representation π such that $\pi^{K(1)}$ is multiplicity free.

The theorem is obtained by combining results of [**EGS15**] with a description of the submodule structure of generic $\mathrm{GL}_2(\mathbf{F}_q)$-projective envelopes. Note that this theorem precludes infinitely many representations constructed in the proof of [**Hu10**, Theorem 4.17] from appearing in completed cohomology.

3. The GL_3 picture

In this section we consider the mod p local Langlands correspondence for $\mathrm{GL}_3(\mathbf{Q}_p)$ in more detail. We are particularly interested in the smooth mod p representations π which appear in Hecke eigenspaces for the mod p cohomology of $U(3)$ arithmetic manifolds with infinite level at p. In particular we introduce modifications of the classical U_p-operators which let us determine the local Galois parameter attached to such a π.

We now fix the general setting to state our main results.

Let F/F^+ be a CM extension of number fields and fix a place $w|p$. We fix a unitary group $G_{/F^+}$ such that $G \times F \cong \mathrm{GL}_3$ and $G(F_v^+) \cong U_3(\mathbf{R})$ for all $v|\infty$. Choose a model $\mathcal{G}_{/\mathscr{O}_{F^+}}$ of G such that $\mathcal{G} \times \mathscr{O}_{F_{v'}^+}$ is reductive for all places v' of F^+ that split in F. Let $v = w|_{F^+}$. Choose a compact open subgroup $U = U^v \times \mathcal{G}(\mathscr{O}_{F_v^+}) \leq G(\mathbf{A}_{F^+}^\infty)$ that is sufficiently small, and unramified at all places dividing p. Let V' denote an irreducible smooth representation of $\prod_{v'|p, v' \neq v} \mathcal{G}(\mathscr{O}_{F_{v'}^+})$ over $\mathbf{F}$ determined by a highest weight in the lowest alcove, and let $\widetilde{V}'$ denote a Weyl module of $\prod_{v'|p, v' \neq v} \mathcal{G}(\mathscr{O}_{F_{v'}^+})$ over $\mathscr{O}$ of the same highest weight, so $\widetilde{V}' \otimes_{\mathscr{O}} \mathbf{F} \cong V'$. (For simplicity one can assume that $V' = \mathbf{F}$ and $\widetilde{V}' = \mathscr{O}$.) We define in the usual way spaces of mod p automorphic forms $S(U^v, V') = \varinjlim_{U_v \leq G(F_v^+)} S(U^v U_v, V')$ and similarly $S(U^v, \widetilde{V}')$ that are smooth representations of $G(F_v^+) \cong \mathrm{GL}_3(\mathbf{Q}_p)$ (where this isomorphism depends on our chosen place $w|v$).

We fix a cofinite subset $\mathcal{P}$ of places $w' \nmid p$ of F that split over F^+, such that U is unramified at $w'|_{F^+}$, and such that $\overline{r}$ is unramified at w'. Then the abstract Hecke algebra $\mathbf{T}^{\mathcal{P}}$ generated over $\mathscr{O}$ by Hecke operators at all places in $\mathcal{P}$ acts on $S(U^v, V')$ and $S(U^v, \widetilde{V}')$, commuting with the $\mathrm{GL}_3(\mathbf{Q}_p)$-actions.

Let now $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ be an absolutely irreducible Galois representation. Then $\overline{r}$ determines a maximal ideal $\mathfrak{m}_{\overline{r}}$ of $\mathbf{T}^{\mathcal{P}}$. We assume that $\overline{r}$ is automorphic in this setup, meaning that $S(U^v, V')[\mathfrak{m}_{\overline{r}}] \neq 0$ (or equivalently, $S(U^v, V')_{\mathfrak{m}_{\overline{r}}} \neq 0$).

From now until the end of this section *we assume that p splits completely in F* (in particular $F_w \cong \mathbf{Q}_p$).

3.1. GL_3 local-global compatibility: ordinay case.

— We assume now that $\overline{r}|_{G_{F_w}}$ is upper-triangular, maximally non-split, and generic. This means that

$$\overline{r}|_{G_{F_w}} \sim \begin{pmatrix} \omega^{a+1}\mathrm{un}_{\mu_2} & *_1 & * \\ & \omega^{b+1}\mathrm{un}_{\mu_1} & *_2 \\ & & \omega^{c+1}\mathrm{un}_{\mu_0} \end{pmatrix}, \tag{3.1}$$

the extensions $*_1$, $*_2$ are non-split, and $a-b>2$, $b-c>2$, $a-c<p-3$. (Here, ω is the mod p cyclotomic character and un_μ denotes the unramified character taking value $\mu \in \mathbf{F}^\times$ on geometric Frobenius elements.) Once the diagonal characters are fixed, the isomorphism class of $\overline{r}|_{G_{F_w}}$ is determined by an invariant $\mathrm{FL}(\overline{r}|_{G_{F_w}})$ which is defined by means of Fontaine-Laffaille theory, and which can take any value in $\mathbb{P}^1(\mathbf{F}) \setminus \{\mu_1\}$. Work of Breuil and Herzig [**BH15**] shows that the $\mathrm{GL}_3(\mathbf{Q}_p)$-representation $S(U^v, V')[\mathfrak{m}_{\overline{r}}]$ determines the ordered triple of diagonal characters $(\omega^{a+1}\mathrm{un}_{\mu_2}, \omega^{b+1}\mathrm{un}_{\mu_1}, \omega^{c+1}\mathrm{un}_{\mu_0})$ of $\overline{r}|_{G_{F_w}}$. In fact, the triple (a,b,c) can be recovered from the (ordinary part of the) $\mathrm{GL}_3(\mathbf{Z}_p)$-socle by [**GG12**] and the $\mu_i \in \mathbf{F}^\times$ are determined by the Hecke action at p on the $\mathrm{GL}_3(\mathbf{Z}_p)$-socle. It therefore remains for us to show that $S(U^v, V')[\mathfrak{m}_{\overline{r}}]$ determines the invariant $\mathrm{FL}(\overline{r}|_{G_{F_w}})$.

Let I denote the Iwahori subgroup of $\mathrm{GL}_3(\mathbf{Z}_p)$. If V is a representation of $\mathrm{GL}_3(\mathbf{Z}_p)$ over $\mathscr{O}$ and $a_i \in \mathbf{Z}$ we write

$$V^{I,(a_2,a_1,a_0)} \stackrel{\text{def}}{=} \mathrm{Hom}_I(\mathscr{O}(\widetilde{\omega}^{a_2} \otimes \widetilde{\omega}^{a_1} \otimes \widetilde{\omega}^{a_0}), V),$$

where the character in the domain denotes the inflation to I of the homomorphism $\mathrm{B}(\mathbf{F}_p) \to \mathscr{O}^\times$, $\begin{pmatrix} x & y & z \\ & u & v \\ & & w \end{pmatrix} \mapsto \widetilde{x}^{a_2}\widetilde{u}^{a_1}\widetilde{w}^{a_0}$. If V is even a representation of $\mathrm{GL}_3(\mathbf{Q}_p)$, then $V^{I,(a_2,a_1,a_0)}$ affords an action of U_p-operators U_1, U_2.

Define $\Pi \stackrel{\text{def}}{=} \begin{pmatrix} & 1 & \\ & & 1 \\ p & & \end{pmatrix}$ (it sends $V^{I,(a_2,a_1,a_0)}$ to $V^{I,(a_1,a_0,a_2)}$).

We now introduce explicit group algebra operators S, $S' \in \mathbf{F}[\mathrm{GL}_3(\mathbf{F}_p)]$, defined on $S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I,(-b,-c,-a)}[U_1, U_2]$, which will play a crucial role in determining the local Galois parameter from $S(U^v, V')[\mathfrak{m}_{\overline{r}}]$:

$$\begin{aligned} S &\stackrel{\text{def}}{=} \sum_{x,y,z \in \mathbf{F}_p} x^{p-(a-c)} z^{p-(a-b)} \begin{pmatrix} y & x & 1 \\ z & 1 & \\ 1 & & \end{pmatrix} \\ S' &\stackrel{\text{def}}{=} \sum_{x,y,z \in \mathbf{F}_p} x^{p-(b-c)} z^{p-(a-c)} \begin{pmatrix} y & x & 1 \\ z & 1 & \\ 1 & & \end{pmatrix} \end{aligned} \tag{3.2}$$

The first main theorem is:

Theorem 3.1 ([**HLM17**]). — *We make the following additional assumptions:*

1. $\mathrm{FL}(\overline{r}|_{G_{F_w}}) \notin \{0, \infty\}$.
2. *The $\mathscr{O}$-dual of $S(U^v, \widetilde{V}')^{I,(-b,-c,-a)}_{\mathfrak{m}_{\overline{r}}}$ is free over $\mathbf{T}$, where $\mathbf{T}$ denotes the $\mathscr{O}$-subalgebra of $\mathrm{End}\big(S(U^v, \widetilde{V}')^{I,(-b,-c,-a)}_{\mathfrak{m}_{\overline{r}}}\big)$ generated by $\mathbf{T}^{\mathcal{P}}$, U_1, and U_2.*

Then we have the equality

$$S' \circ \Pi = (-1)^{a-b} \cdot \frac{b-c}{a-b} \cdot \mathrm{FL}(\overline{r}|_{G_{F_w}}) \cdot S$$

of maps

$$S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I,(-b,-c,-a)}[U_1, U_2] \to S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I,(-c-1,-b,-a+1)}.$$

Moreover, these maps are injective with non-zero domain. In particular, $\mathrm{FL}(\overline{r}|_{G_{F_w}})$ *is determined by the smooth* $\mathrm{GL}_3(\mathbf{Q}_p)$*-representation* $S(U^v, V')[\mathfrak{m}_{\overline{r}}]$.

The first assumption is related to the fact that the $\mathrm{GL}_3(\mathbf{Z}_p)$-socle of $S(U^v, V')[\mathfrak{m}_{\overline{r}}]$ changes for the exceptional two invariants, see Theorem 3.4 below.

We show that the second assumption is often a consequence of the first assumption. The second assumption is an analogue of Mazur's "mod p multiplicity one" result:

Theorem 3.2 **([HLM17])**. — *Let $\overline{r}$ be as in Theorem 3.1(i). Assume $\overline{r}$ satisfies some standard Tailor-Wiles conditions. If we have* $\mathrm{FL}(\overline{r}|_{G_{F_w}}) \neq \infty$*, then assumption (ii) in Theorem 3.1 holds.*

In fact for any value of $\mathrm{FL}(\overline{r}|_{G_{F_w}})$ either the second assumption or its dual holds. For sake of completeness we state the standard Taylor-Wiles conditions mentionned in Theorem 3.2:

(i) F/F^+ is unramified at all finite places.
(ii) $\overline{r}$ is unramified at all finite places not dividing p;
(iii) $\overline{r}$ is Fontaine–Laffaille and regular at all places dividing p;
(iv) $\overline{r}$ has image containing $\mathrm{GL}_3(k)$ for some $k \subseteq \mathbf{F}$ with $\#k > 9$;
(v) $\overline{F}^{\ker \mathrm{ad}\overline{r}}$ does not contain $F(\zeta_p)$;
(vi) $U_{v'} = \mathcal{G}(\mathscr{O}_{F^+_{v'}})$ for all places v' which split in F other than v_1 and those dividing p (here $v_1 \nmid p$ is a finite place of F^+, splitting as $v_1 = w_1 w_1^c$ in F and such that v_1 does not split completely in $F(\zeta_p)$ and $\overline{r}(\mathrm{Frob}_{w_1})$ has distinct $\mathbf{F}$-rational eigenvalues, no two of which have ratio $(\mathbf{N}_{F^+/\mathbf{Q}}(v_1))^{\pm 1}$);
(vii) U_{v_1} is the preimage of the upper-triangular matrices under the map

$$\mathcal{G}(\mathscr{O}_{F^+_{v_1}}) \to \mathcal{G}(\mathbf{F}_{v_1}) \xrightarrow[\iota_{w_1}]{\sim} \mathrm{GL}_3(\mathbf{F}_{w_1});$$

(viii) $U_{v'}$ is a hyperspecial maximal compact open subgroup of $G(F^+_{v'})$ if v' is inert in F.

Using results of [**EG14**], we see that for any given local Galois representation as in (3.1) we can construct a globalization to which Theorem 3.2 applies.

Theorem 3.3 **([HLM17])**. — *Suppose that $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ is upper-triangular, maximally non-split, and generic. Then, after possibly replacing $\mathbf{F}$ by a finite extension, there exist a CM field F, a Galois representation $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$, a place $w|p$ of F, groups $G_{/F^+}$ and $\mathcal{G}_{/\mathscr{O}_{F^+}}$, and a compact open subgroup U^v such that $\overline{r}|_{G_{F_w}} \cong \overline{\rho}$. In particular, if* $\mathrm{FL}(\overline{\rho}) \notin \{0, \infty\}$*, Theorem 3.1 applies to $\overline{r}$.*

As a by-product of our methods we almost completely determine the set of Serre weights of $\overline{r}$. Here, the set $W_w(\overline{r})$ is defined to be the set of irreducible $\mathrm{GL}_3(\mathbf{Z}_p)$-representations whose duals occur in the $\mathrm{GL}_3(\mathbf{Z}_p)$-socle of $S(U^v, V')_{\mathfrak{m}_{\overline{r}}}$ (for some U^v and $\mathcal{P}$ as above).

Theorem 3.4 (**[HLM17]**). — *Keep the assumptions on $\overline{r}$ that precede Theorem 3.1 above.*

1. *If* $\mathrm{FL}(\overline{r}|_{G_{F_w}}) \notin \{0, \infty\}$ *we have*
$$\{F(a-1, b, c+1)\} \subseteq W_w(\overline{r}) \subseteq$$
$$\subseteq \{F(a-1, b, c+1),\ F(c+p-1, b, a-p+1)\}.$$
2. *If* $\mathrm{FL}(\overline{r}|_{G_{F_w}}) = \infty$ *we have*
$$\{F(a-1, b, c+1), F(a, c, b-p+1)\} \subseteq W_w(\overline{r}) \subseteq$$
$$\subseteq \{F(a-1, b, c+1),\ F(c+p-1, b, a-p+1), F(a, c, b-p+1)\}.$$
3. *If* $\mathrm{FL}(\overline{r}|_{G_{F_w}}) = 0$ *we have*
$$\{F(a-1, b, c+1), F(b+p-1, a, c)\} \subseteq W_w(\overline{r}) \subseteq$$
$$\subseteq \{F(a-1, b, c+1),\ F(c+p-1, b, a-p+1), F(b+p-1, a, c)\}.$$

The dependence on $\mathrm{FL}(\overline{r}|_{G_{F_w}})$ in this theorem was unexpected, as there were no explicit Serre weight conjectures in the literature that apply to non-semisimple $\overline{r}|_{G_{F_w}}$.

As a consequence of this theorem we also show the existence of an automorphic lift r of $\overline{r}$ such that $r|_{G_{F_w}}$ is upper-triangular. It is in the same spirit as the main results of **[BLGG13]** (which concerned two-dimensional representations).

Theorem 3.5 (**[HLM17]**). — *In the setting of Theorem 3.4, $\overline{r}$ has an automorphic lift $r : G_F \to \mathrm{GL}_3(\mathscr{O})$ (after possibly enlarging E) such that $r|_{G_{F_w}}$ is crystalline and ordinary of Hodge–Tate weights $\{-a-1, -b-1, -c-1\}$.*

3.1.1. Brief overview of the methods. — Theorems 3.1 and 3.2 generalize earlier work of Breuil–Diamond **[BD14]** which treated two-dimensional Galois representations of G_F, where F is totally real and p is unramified in F. The strategy in the proof of Theorem 3.2 is similar: we lift the Hecke eigenvalues of $\overline{r}$ to a well-chosen type in characteristic zero, use classical local-global compatibility at p, and then study carefully how both the Galois-side and the GL_3-side reduce modulo p. However, it is significantly more difficult to carry out this strategy in the GL_3-setting.

We first prove the upper bound in Theorem 3.4 by lifting to various types in characteristic zero and using integral p-adic Hodge theory to reduce modulo p. This is more involved in dimension 3, since we are no longer in the potentially

Barsotti–Tate setting. We crucially use results of Caruso to filter our Breuil module (corresponding to $\overline{r}|_{G_{F_w}}$) according to the socle filtration on $\overline{r}|_{G_{F_w}}$.

The following theorem is our key local result on the Galois-side. Our chosen type is a tame principal series that contains in its reduction mod p all elements of $W_w(\overline{r})$ (unlike in [**BD14**], where the intersection always consisted of one element) and we obtain a rough classification of the strongly divisible module corresponding to ρ, in particular their "Hodge" filtration. Note that we do not need to determine Frobenius and monodromy operators: in the language of [**LLHLMb**] we are considering universal finite height lift of $\overline{r}|_{G_{F_w}}$ in shape $\alpha\beta\alpha$. We also note that the relevant information on the Galois-side is independent of the Hodge filtration, so that we can transfer this information to the GL_3-side using classical local-global compatibility.

Theorem 3.6 **([HLM17])**. — *Let $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathscr{O})$ be a potentially semistable p-adic Galois representation of Hodge–Tate weights $\{-2,-1,0\}$ and inertial type $\widetilde{\omega}^a \oplus \widetilde{\omega}^b \oplus \widetilde{\omega}^c$. Assume that the residual representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ is upper-triangular, maximally non-split, and generic as in* (3.1)*. Let $\lambda \in \mathscr{O}$ be the Frobenius eigenvalue on $\mathrm{D}_{\mathrm{st}}^{\mathbf{Q}_p,2}(\rho)^{I_{\mathbf{Q}_p}=\widetilde{\omega}^b}$. Then the Fontaine–Laffaille invariant of $\overline{\rho}$ is given by:*

$$\mathrm{FL}(\overline{\rho}) = \mathrm{red}\left(p\lambda^{-1}\right),$$

where red *denotes the specialization map $\mathbb{P}^1(\mathscr{O}) \to \mathbb{P}^1(\mathbf{F})$.*

On the GL_3-side the main innovation consists of the explicit group algebra operators S, S' defined in (3.2). The analogues of these operators for GL_2 show up in various contexts (see, for example, [**Pas07**], Lemma 4.1, [**BP12**], Lemma 2.7, [**Bre11**], §4, and [**BD14**], Proposition 2.6.1). .

Proposition 3.7 **([HLM17])**. —

1. *There is a unique non-split extension of irreducible $\mathrm{GL}_3(\mathbf{F}_p)$-representations*
$$0 \to F(-c-1,-b,-a+1) \to V \to F(-b+p-1,-c,-a) \to 0$$
and S induces an isomorphism $S : V^{I,(-b,-c,-a)} \xrightarrow{\sim} V^{I,(-c-1,-b,-a+1)}$ of one-dimensional vector spaces.
2. *There is a unique non-split extension of irreducible $\mathrm{GL}_3(\mathbf{F}_p)$-representations*
$$0 \to F(-c-1,-b,-a+1) \to V \to F(-c,-a,-b-p+1) \to 0$$
and S' induces an isomorphism $S' : V^{I,(-c,-a,-b)} \xrightarrow{\sim} V^{I,(-c-1,-b,-a+1)}$ of one-dimensional vector spaces.

The reduction mod p result on the GL_3-side is comparatively easier. By combining the above results we deduce Theorem 3.1. We note that assumption (ii) is needed for lifting elements of $S(U^v, V')[\mathfrak{m}_{\overline{r}}]^{I,(-b,-c,-a)}[U_1, U_2]$ to suitable Iwahori eigenvectors in characteristic zero. The U_i-operators allow us to deal with the possible presence of the shadow weight $F(c+p-1,b,a-p+1)$ in Theorem 3.4. Namely, if $v\in S(U^v,V')[\mathfrak{m}_{\overline{r}}]^{I,(-b,-c,-a)}[U_1,U_2]$ is non-zero, we show that it generates the representation of Proposition 3.7(i) under the $\mathrm{GL}_3(\mathbf{Z}_p)$-action. Similar comments apply to Πv. Proposition 3.7 then allows us to deduce that the maps in Theorem 3.1 are well-defined and injective.

Finally, we establish Theorem 3.2. As in [**BD14**] our method relies on the Taylor–Wiles method. However, as we do not know whether our local deformation ring at p is formally smooth (which in any case should be false if our chosen type intersects $W_w(\overline{r})$ in more than one element) we cannot directly apply Diamond's method [**Dia97**]. Instead we use the patched modules of [**CEG+16**] that live over the universal deformation space at p and use ideas of [**EGS15**] and [**Le**]. Similarly to above, we add U_p-operators in order to deal with the possible presence of the shadow weight $F(c+p-1,b,a-p+1)$.

3.2. GL_3 **local-global compatibility: non-ordinay case.** — We keep the very same notations and setting of the previous section, but we now assume that $\overline{r}|_{G_{F_w}}$ has niveau 2. This means that

$$\overline{r}|_{I_{F_w}}\sim\begin{pmatrix}\omega^{a+1} & * & *\\ 0 & \omega_2^{(b+1)+p(c+1)} & 0\\ 0 & 0 & \omega_2^{c+1+p(b+1)}\end{pmatrix}\tag{3.3}$$

for some integers $a,b,c\in\mathbf{N}$ satisfying $p-3>a-c-1>b-c>3$. If $\overline{\rho}$ is Fontaine-Laffaille we can define an invariant $\mathrm{FL}(\overline{r}|_{G_{F_w}})\in\mathbb{P}^1(\mathbf{F})$ generalizing the one in the previous section.

Quite surprisingly, we can recover the Fontaine-Laffaille parameter from the Hecke action on $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$ by using exactly the same group algebra operator we used in the ordinary case.

The following Theorem is the non-ordinary analogue of Theorems 3.1, 3.2 above

Theorem 3.8 **([LMP])**. — *Assume that* $\mathrm{FL}(\overline{r}|_{G_{F_w}})\notin\{0,\infty\}$ *and that* $\overline{r}$ *satisfies the standard Taylor-Wiles conditions (i)-(viii) above.*

Let S,S' *be the group algebra operators defined in (3.2). Then*

$$S'\circ\begin{pmatrix}0&1&0\\0&0&1\\p&0&0\end{pmatrix}=(-1)^{a-b}\cdot\frac{b-c}{a-b}\cdot\mathrm{FL}(\overline{r}|_{G_{F_w}})\cdot S\tag{3.4}$$

on $S(U^v,V')[\mathfrak{m}_{\overline{r}}]^{I,(-b,-c,-a)}[U_2]$, *where is a carefully chosen* U_p*-operator.*

As it happened in the ordinary case, in order to get Theorem 3.8 one needs multiplicity one conditions on the $G(\mathcal{O}_{F_v^+})$-socle of $S(U^v,V')[\mathfrak{m}_{\overline{r}}]$. This is obtained by a thorough *type elimination* in niveau 2, which highlights again that the set of Serre weights for $\overline{r}$ depends on the associated Fontaine-Laffaille parameter.

We define here an explicit set $W^?_w(\overline{r}|_{G_{F_w}})$, depending on $\mathrm{FL}(\overline{r}|_{G_{F_w}})$ as follows:

$$W^?_w(\overline{r}) = \left\{ \begin{array}{c} F(a-1,b,c+1),\ \ F((p-1)+c,b,a-(p-1)), \\ F(a-1,c+1,b-(p-1)) \end{array} \right\} \bigcup W$$

where

$$W \stackrel{\text{def}}{=} \begin{cases} \left\{ \begin{array}{c} F(p-1+c,a,b),\ F(p-2+b,a,c+1), \\ F(a,c,b-(p-1)) \end{array} \right\} & \text{if } \mathrm{FL}(\overline{r}|_{G_{F_w}}) = \infty; \\ \\ \left\{F((p-1)+b,a,c)\right\} & \text{if } \mathrm{FL}(\overline{r}|_{G_{F_w}}) = 0; \\ \\ \emptyset & \text{otherwise.} \end{cases}$$

We remark that in the set $W^?_w(\overline{r}|_{G_{F_w}})$ we can distinguish an explicit subset $W^?_{w,\mathrm{obv}}(\overline{r})$ of obvious weights related to "obvious" crystalline lifts of $\overline{r}|_{G_{F_w}}$.

Our main result on Serre weights for $\overline{r}$ is contained in the following theorem:

Theorem 3.9 (**[LMP]**). — *Assume that $\overline{r}$ verifies assumption (i) of Theorem 3.8. Then*

$$W_w(\overline{r}) \subseteq W^?_w(\overline{r}|_{G_{F_w}}).$$

Moreover, the obvious *weights $F(a-1,b,c+1)$ and $F(a-1,c+1,b-p+1)$ are always modular, while, if the Fontaine-Laffaille parameter at w verifies $\mathrm{FL}(\overline{r}|_{G_{F_w}}) = \infty$, the* shadow *weight $F(a,c,b-(p-1))$ is modular.*

Finally, assume that F is unramified at all finite places and that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbf{A}_F)$ of level prime to p such that

1. *$\overline{r} \simeq \overline{r}_{p,\imath}(\Pi)$;*
2. *For each place $w|p$ of F, $r_{p,\imath}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable;*
3. *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Then we have the following inclusion:

$$W^?_{w,\mathrm{obv}}(\overline{r}) \subseteq W_w(\overline{r}).$$

(Here we wrote $\overline{r}_{p,i}(\Pi)$ to denote the Galois representation associated to Π as in [**BLGG**], Theorem 2.1.2; it depends on the choice of an isomorphism $\imath : \overline{\mathbf{Q}}_p \to \mathbf{C}$.)

For the weight elimination results, we classify rank 2 simple Breuil modules with descent data of niveau 2 corresponding to the 2-dimensional irreducible quotient of $\overline{r}|_{G_{F_w}}$. The classification of the rank 2 simple Breuil modules is also heavily used to show the connection between the Fontaine-Laffaille parameter and Frobenius eigenvalues of potentially crystalline lifts of $\overline{r}|_{G_{F_w}}$. The proof of weight existence is here performed by purely Galois cohomology arguments. We remark that along the proof of Theorem 3.9, we obtain a potential diagonalizability result, which lets us infer that representations satisfying the hypotheses of Theorem 3.8 do exist:

Proposition 3.10 ([**LMP**]). — *Let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ be a niveau 2 continuous Galois representation as in (3.3). Then $\overline{\rho}$ admits a crystalline lift $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathscr{O})$ such that $\rho|_{G_{\mathbf{Q}_{p^2}}}$ is ordinary crystalline, with parallel Hodge-Tate weights $\{a+1, b+1, c+1\}$. In particular ρ is potentially diagonalizable.*

Moreover, if $\mathrm{FL}(\overline{\rho}) = \infty$ then $\overline{\rho}$ admits a crystalline lift with Hodge-Tate weights $\{p+c+1, a+1, b\}$.

Finally if $\overline{\rho}$ is split then then $\overline{\rho}$ admits a further crystalline lift with Hodge-Tate weights $\{p+b, p+c, a+1\}$.

3.3. An explicit set of weights in the wild case. — We keep the notations and the setting of §3.1 . In this section we examine the set of modular weights of $\overline{r}|_{G_{F_w}}$ when the latter is ordinary, wildly ramified, and has radical length exactly 2. This generalizes Theorem 3.4 above, and provides the first systematic description of the set of modular weights for wildly ramified, three dimensional representations of $G_{\mathbf{Q}_p}$. The submodule structure of $\overline{r}|_{G_{F_w}}$ plays now a crucial role (as in [**BH15**]) and we define, for all possible configurations of the submodule structure in $\overline{r}|_{G_{F_w}}$, a set of weights $W^?_w(\overline{r}|_{G_{F_w}})$ for which $\overline{r}$ shall be modular.

Up to duality, we can distinguish 3 isomorphism classes for $\overline{r}|_{I_{F_w}}$:

Type (T_2) if

$$\overline{r}|_{I_{F_w}} : \qquad \begin{array}{ccc} & & \omega^{b+1} \\ \omega^{a+1} & \begin{array}{c} \diagup \\ \diagdown \end{array} & \\ & & \omega^{c+1} \end{array}$$

Type (T_1) if

$$\overline{r}|_{I_{F_w}} : \qquad \begin{array}{c} \omega^{a+1} \;\text{——}\; \omega^{b+1} \\ \oplus \\ \omega^{c+1} \end{array}$$

Type (T_0) if

$$\overline{r}|_{I_{F_w}} : \qquad \begin{array}{c} \omega^{a+1} \;\text{——}\; \omega^{c+1} \\ \oplus \\ \omega^{b+1} \end{array}$$

Accordingly, we define the following set of Serre weights:

$$W_w^?(\overline{r}|_{G_{F_w}}) \overset{\text{def}}{=} \big\{F(a-1,b,c+1),\ F(c+(p-1),b,a-(p-1))\big\} \cup W$$

where

if $\overline{r}|_{I_{F_w}}$ is of Type (T_2)

$$W := \big\{F(a-1,c,(b+1)-(p-1))\big\};$$

if $\overline{r}|_{I_{F_w}}$ is of Type (T_1)

$$W \overset{\text{def}}{=} \left\{ \begin{array}{cc} F(a-1,c,b+1-(p-1)), & F(c-1+(p-1),a,b+1) \\ F(b+(p-1),a,c), & F(a,c,b-(p-1)) \end{array} \right\};$$

if $\overline{r}|_{I_{F_w}}$ is of Type (T_0)

$$W \overset{\text{def}}{=} \big\{F(a-1,c,(b+1)-(p-1)),\quad F(b-1+(p-1),a,c+1)\big\};$$

The main result of this section is the following:

Theorem 3.11 **([MP])**. — *Keep the hypotheses and setting above, in particular the Galois parameter $\overline{r}|_{G_{F_w}}$ being ordinary generic, and of Loewy length 2.*

Then one has

$$W_w(\overline{r}) \subseteq W_w^?(\overline{r}|_{G_{F_w}}).$$

Theorem 3.11 is a weight elimination result and is performed by integral p-adic Hodge theory and classical local/global compatibility. It relies on a crucial ingredient in integral p-adic Hodge theory, namely a splitting lemma which lets us determine the vanishing of a class $[c] \in H^1(G_{\mathbf{Q}_p}, \omega^n)$ (ω denoting the mod p-cyclotomic character) by the only information coming from the filtration on a Breuil module $\mathcal{M}$ with descent data such that $[\mathrm{T}_{\mathrm{st}}(\mathcal{M})] = [c]$.

Understanding which weights in $W_w^?(\overline{r}|_{G_{F_w}})$ are actually modular for $\overline{r}$ is on the other hand a hard problem. Following [**BLGG**], automorphy lifting techniques can be used to prove that the set of *obvious weights* $W_{\mathrm{obv}}^?(\overline{r}|_{G_{F_w}})$ are modular, by considering ordinary crystalline lifts of $\overline{r}|_{G_{F_w}}$. One always has $W_{\mathrm{obv}}^?(\overline{r}|_{G_{F_w}}) \subseteq W_w^?(\overline{r}|_{G_{F_w}})$ and by combining Theorem 3.11 and [**BLGG**] Theorem 4.1.9, one obtains the following result:

Theorem 3.12 **([MP])**. — *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places and that p splits completely in F. Suppose that $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ is an irreducible representation with split ramification. Assume that there is a RACSDC automorphic representation Π of $\mathrm{GL}_3(\mathbf{A}_F)$ such that*

- *$\overline{r} \simeq \overline{r}_{p,\imath}(\Pi)$;*
- *For each place $w|p$ of F, $r_{p,\imath}(\Pi)|_{G_{F_w}}$ is potentially diagonalizable;*
- *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Assume further that for all places $w|p$ the Galois representation $\overline{r}|_{G_{F_w}}$ are ordinary, strongly generic, and of Loewy length less than 3.

Then one has

$$W^?_{\mathrm{obv}}(\overline{r}|_{G_{F_w}}) \subseteq W_w(\overline{r}).$$

When p does not split completely in F, the study of Serre weights for wildly ramified, three-dimensional Galois representation is pursued in the forthcoming [**LLHLMd**].

4. Shapes and Shadows

In this section we discuss new techniques to compute potentially crystalline deformation rings of a continuous Galois representation $\overline{\rho} : G_L \to \mathrm{GL}_3(\mathbf{F})$. This has global applications, as new modularity lifting results, the proof of modularity of *shadow* weights, and new cases of the Breuil-Mézard conjecture in dimension three. It also has unexpected applications to some problems in modular representation theory, namely the classification of integral lattices in Deligne-Lusztig representations with $\overline{\mathbf{Q}}_p$-coefficients (§4.3.1).

4.1. Galois deformation rings in the ordinary case. — We compute potentially crystalline deformation rings for an ordinary Galois representation $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ using the theory of strongly divisible modules. The techniques are new, in the sense that we develop a *convergence algorithm* to compute filtrations in strongly divisible modules with descent data in ordinary situations, and we are able to impose Griffiths transversality condition in a simple way. These techniques will be developed in great generality in §4.2.

We explain in more detail the difficulties in integral p-adic Hodge theory which arises in our situation. Let $L/\mathbf{Q}_p$ be a finite extension, which we assume to be totally ramified for simplicity, and let ϖ_L be a uniformizer of L. Choose a compatible system of p-power roots ϖ_L^{1/p^n} and define $L_\infty = \cup_n L(\varpi_L^{1/p^n}) \subset \overline{L}$. Let G_L denote the absolute Galois group of L and let $G_{L_\infty} := \mathrm{Gal}(\overline{L}/L_\infty)$. The restriction functor

$$\mathrm{Rep}^{\mathrm{cris}}_{G_L}(\overline{\mathbf{Q}_p}) \to \mathrm{Rep}_{G_{L_\infty}}(\overline{\mathbf{Q}_p})$$

is fully faithful and its image is contained in the *finite height* G_{L_∞}-representations ([**Kis06**, Corollary 2.1.14]). Both these categories are described by linear algebra data using p-adic Hodge theory. From that perspective, the essential image of the functor is characterized by a Griffiths transversality condition. In the Barsotti-Tate case (i.e., height ≤ 1), Griffiths transversality is always satisfied and this gives more precise control over Barsotti-Tate deformation rings. The difficulty that arises in higher weight situations is to understand which representations of G_{L_∞} descend to (potentially) crystalline representations of G_L (integrally as well). In this section, we address this question for tamely potentially crystalline deformation rings for $\mathbf{Q}_p$ and GL_3 with Hodge-Tate weights $(0,1,2)$ with some assumptions on $\overline{\rho}$.

In order to state the main theorem, we let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ be an ordinary three dimensional Galois representation of the form

$$\overline{\rho}|_{I_{\mathbf{Q}_p}} \cong \begin{pmatrix} \omega^{a+2} & * & * \\ 0 & \omega^{b+1} & * \\ 0 & 0 & \omega^c \end{pmatrix} \tag{4.1}$$

where $\omega : I_{\mathbf{Q}_p} \to \mathbf{F}_p$ denotes the mod p cyclotomic character and $a, b, c \in \mathbf{N}$. Recall that $\widetilde{\omega} : I_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$ denotes the Teichmüller lift of ω.

Theorem 4.1 (**[LM16]**). — *Let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ be an ordinary Galois representation as in (4.1). Assume that the integers $a, b, c \in \mathbf{N}$ verify $b-c$, $a-b > 3$ and $a-c < p-4$ and define the inertial type $\tau \overset{def}{=} \widetilde{\omega}^a \oplus \widetilde{\omega}^b \oplus \widetilde{\omega}^c$.*

Let $R_{\overline{\rho}}^{\square,(0,1,2)\tau}$ be the framed potentially crystalline deformation ring for $\overline{\rho}$, with Galois type τ and Hodge type $(0,1,2)$ and assume that $\mathrm{Spf} R_{\overline{\rho}}^{\square,(0,1,2),\tau}$ is non-empty. Then $R_{\overline{\rho}}^{\square,(0,1,2),\tau}$ is formally smooth of relative dimension 12.

Theorem 4.1 is obtained by explicitly constructing a formally smooth morphism $R_{\overline{\mathcal{M}}}^{\square,\tau} \to R_{\overline{\rho}}^{\square,(0,1,2),\tau}$, where $R_{\overline{\mathcal{M}}}^{\square,\tau}$ is a moduli space of strongly divisible modules $\mathcal{M}$ lifting $\overline{\rho}$ which we can control by means of integral p-adic Hodge theory (and $\overline{\mathcal{M}}$ is a Breuil module giving rise to $\overline{\rho}$). There are two key ingredients. First, for our choice of τ, a detailed study of the filtration and Frobenius building on techniques of [**Bre14**] shows that any strongly divisible module lifting $\overline{\rho}$ is ordinary. This uses a convergence algorithm in the Breuil ring $S_{\mathbf{Q}_p}$ which lets us compute the filtration on strongly divisible modules lifting $\overline{\mathcal{M}}$. The algorithm is inspired by [**Bre14**], Proposition 5.2, where we find a classification of filtrations on strongly divisible modules in the potentially Barsotti-Tate case.

We thus obtain a formally smooth family of ordinary quasi-Breuil modules (i.e., with no monodromy operator). Now as a consequence of the genericity assumptions on τ the condition imposed by the existence of monodromy on an ordinary rank 3 Breuil module mod p turns out to be exceedingly simple. In this case, the vanishing of a single variable of the smooth family of quasi-Breuil modules.

Theorem 4.1 is compatible with the Breuil-Mézard conjecture. In the hypotheses of Theorem 4.1, the inertial type τ contains exactly one weight in the conjectural set of Serre weights for $\overline{\rho}$; it is an obvious weight for $\overline{\rho}$ in the terminology of [**GHS**], in the Fontaine-Laffaille range. In particular, the Breuil-Mézard conjecture then predicts that $R_{\overline{\rho}}^{\square,(0,1,2),\tau}$ should be formally smooth and so Theorem 4.1 confirms an instance of the conjecture for GL_3. Moreover, while proving Theorem 4.1, we also explicitly exhibit the geometric Breuil-Mézard conjecture of [**EG14**] in this setting. Namely, we show that the special fiber of $\mathrm{Spf} R_{\overline{\rho}}^{\square,(0,1,2),\tau}$ inside the unrestricted universal framed Galois deformation space coincides with the special fiber of the (Fontaine-Laffaille) crystalline deformation ring with Hodge-Tate weights $(a+2, b+1, c)$.

As a consequence of our careful study of the filtration and Frobenius on strongly divisible modules, we get the following corollary:

Theorem 4.2 (**[LM16]**). — *Let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$ be an ordinary Galois representation as in (4.1). Assume that the inertial type τ is as in Theorem 4.1 and that the integers a, b, c verify $p - 4 > a - b, b - c > 3$. Then, any potentially crystalline lift ρ of $\overline{\rho}$, with Hodge type $(0, 1, 2)$ and inertial type τ is ordinary.*

We remark that for Theorem 4.2, we do not require a Fontaine-Laffaille condition on the inertial weights.

4.2. Deformation rings: Shapes and Shadows. — This section builds on [**LLHLMb**]. We discuss new techniques to compute potentially crystalline deformation rings in dimension three and their connection to Serre-weight and Breuil-Mézard conjectures.

Connections between the weight part of Serre's conjecture and the geometrization of the Breuil-Mézard conjecture ([**BM02**], [**BM14**], [**EG14**], [**EG**]) appear first in the breakthrough work [**GK14**]. A key insight of the geometric Breuil-Mézard conjecture is the prediction of the multiplicities of the special fibers of deformation spaces (or, more generally, moduli stacks) of local Galois representations when $\ell = p$ in terms of p-modular representations of general linear groups. In particular, a good understanding of the geometry of local Galois deformation spaces leads naturally to modularity lifting results, Breuil-Mézard and the weight part of Serre's conjecture, via the patching techniques of Kisin-Taylor-Wiles.

In dimension two, potentially Barsotti-Tate (BT) deformation rings were studied via moduli of finite flat group schemes ([**Kis09**], [**Bre00**]) leading to explicit presentations when $L/\mathbf{Q}_p$ is unramified ([**BM14, EGS15**]). The geometry of these (potentially) BT-deformation rings is a key input into the proof of the weight part of Serre's conjectures in [**GK14**] and provided evidence towards mod-p local Langlands. However, a satisfactory understanding of the n-dimensional analogue, potentially crystalline deformation rings with Hodge-Tate weights $(n-1, n-2, \ldots, 0)$, seemed out of reach, due to the difficulty of understanding the monodromy operator in the theory of Breuil-Kisin modules.

In this section we overcome this difficulty in dimension 3 and achieve an almost complete description of the local deformation rings $R_{\overline{\rho}}^{(2,1,0),\tau}$ for $L/\mathbf{Q}_p$ unramified and τ a generic tame inertial type. We thereby obtain the first examples in dimension greater than 2 of Galois deformation rings which are neither ordinary nor Fontaine-Laffaille. Our results are consistent with the Breuil-Mézard conjecture and lead to improvements in modularity lifting.

4.2.1. Results on local deformation spaces.— Let $L/\mathbf{Q}_p$ be a finite unramified extension. We fix a sufficiently large finite extension $E/\mathbf{Q}_p$, $\mathcal{O}$ its ring of integers and $\mathbf{F}$ its residue field (the *rings of coefficients* for our representations).

Let $\tau : I_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathscr{O})$ be a tame inertial type and $\overline{\rho} : G_L \to \mathrm{GL}_3(\mathbf{F})$ be a continuous Galois representation. We introduce a notion of *genericity* for $\overline{\rho}$ (resp. for τ), which is a mild condition on the inertial weights of $\overline{\rho}$ (resp. of τ). Our main local results are a detailed description of framed potentially crystalline deformation ring $R_{\overline{\rho}}^{(2,1,0),\tau}$ in terms of the notion of *shape* attached to the pair $(\overline{\rho}, \tau)$. The *shape* is an element in the Iwahori-Weyl group of GL_3 of length ≤ 4, and arises from the study of moduli of Kisin modules with descent datum (inspired by work of [**Bre14, BM14, CDM, EGS15**] and further pursued in [**CL**]); it generalizes the notion of *genre* which is crucial in [**Bre14**] in describing tamely Barsotti-Tate deformation rings for GL_2.

Theorem 4.3 ([**LLHLMb**]). — *Let $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(\mathbf{F})$. Let τ be strongly generic tame inertial type. Then the framed potentially crystalline deformation ring $R_{\overline{\rho}}^{(2,1,0),\tau}$ with Hodge-Tate weights $(2,1,0)$ has connected generic fiber and its special fiber is as predicted by the geometric Breuil-Mézard conjecture.*

If the shape of $(\overline{\rho}, \tau)$ has length at least 2, then $R_{\overline{\rho}}^{(2,1,0),\tau}$ has an explicit presentation (given in Table 2). If the shape of $(\overline{\rho}, \tau)$ has length ≤ 1, then the special fiber of $R_{\overline{\rho}}^{(2,1,0),\tau}$ has an explicit presentation (described in [**LLHLMb**, §8]*).*

The first step towards Theorem 4.3 is a detailed study of the moduli space of Kisin modules with descent datum. The shapes of Kisin modules which arise as reductions of potentially crystalline representations with Hodge-Tate weights $(2,1,0)$ are indexed by $(2,1,0)$-admissible elements in the Iwahori-Weyl group of GL_3 defined by Kottwitz and Rapoport (cf. [**PZ13**, (9.17)]). For generic τ, the Kisin variety is trivial, and so we can associate a shape to a pair $(\overline{\rho}, \tau)$.

There are 25 elements to be analyzed in $\mathrm{Adm}(2,1,0)$. Due to an additional symmetry, we are able to reduce our analysis to nine cases. The shorter the length of the shape the more complicated the deformation ring is. In seven cases (length ≥ 2), the deformation ring admits a simple description (cf. Table 2 below). The remaining two cases require separate analysis which is undertaken in [**LLHLMb**], §8. Our strategy is as follows:

1. Classify all Kisin modules of shape $\widetilde{w} \in \mathrm{Adm}(2,1,0)$ over $\overline{\mathbf{F}}_p$;
2. For $\overline{\mathfrak{M}}$ of shape $\widetilde{w}$, construct the universal deformation space with height conditions;
3. Impose monodromy condition on the universal family..

Steps (1) and (2) generalize techniques of [**Bre14, CDM, EGS15**] used to compute tamely Barsotti-Tate deformation rings for GL_2. Step (2) amounts to constructing local coordinates for the Pappas-Zhu local model for $(\mathrm{GL}_3, \mu = (2,1,0)$, Iwahori level) (cf. [**CL**]) and requires a more systematic approach to the p-adic convergence algorithm employed by [**Bre14, CDM, LM16**].

Step (3) requires a genuinely new method not present in the tamely Barsotti-Tate case where the link between moduli of finite flat groups schemes and Galois representations is stronger. Kisin [**Kis06**] characterized when a torsion-free Kisin module $\mathfrak{M}$ over $\mathbf{Z}_p$ comes from a crystalline representation in terms of the poles of a monodromy operator $N_{\mathfrak{M}^{\mathrm{rig}}}$ which is naturally defined on the extension $\mathfrak{M}^{\mathrm{rig}}$ of $\mathfrak{M}$ to the rigid analytic unit disk. This condition on the poles of the monodromy operator is a subtle analogue of Griffiths transversality in p-adic Hodge theory. While one cannot compute $N_{\mathfrak{M}^{\mathrm{rig}}}$ completely, it is possible to give an explicit approximation using the genericity condition on τ. The *error term* turns out to be good enough to understand the geometry of the deformation rings.

4.2.2. Global applications.— Using Kisin-Taylor-Wiles patching methods, the local information on the Galois deformation spaces leads to new modularity results and the Serre weight conjectures. To state these results, we fix a global setup and remark that the weight part of Serre's conjecture is expected to be independent of the global setup). Our proofs only use of the existence of *patching functors* in the sense of [**EGS15**], [**GHS**] verifying certain axioms and so our results should carry over to many other situations as well.

We fix the global setup as at the beginning of §3. In particular we have fixed a continuous absolutely irreducible representation $\overline{r} : G_F \to \mathrm{GL}_3(\overline{\mathbf{F}})$, a place v above p, a unitary group G over F^+ which is isomorphic to $U(3)$ at each infinite place and split above p, and an ideal $\mathfrak{m}_{\overline{r}}$ in the anemic Hecke algebra $\mathbf{T}$ acting on the space of mod p algebraic automorphic forms $S(U^v, \mathbf{F})$ with infinite level at v, such that $S(U^v, \mathbf{F})_{\mathfrak{m}_{\overline{r}}} \neq 0$.

In what follows we fix $w|v$ (hence an isomorphism $G(F_v^+) \cong \mathrm{GL}_3(\mathbf{Q}_p)$) and *we keep assuming that p splits completely in F.* (This hypothesis will be removed in §4.3.1)

We let $W_w(\overline{r})$ be the set of irreducible $\mathrm{GL}_3(\mathbf{Z}_p)$-representations whose duals occur in the $\mathrm{GL}_3(\mathbf{Z}_p)$-socle of $S(U^v, \mathbf{F})_{\mathfrak{m}_{\overline{r}}}$. We define a condition of genericity for Serre weights and write $W_w^{gen}(\overline{r})$ for the set of generic modular weights. Namely, if $(a, b, c) \in \mathbf{Z}^3$ is the highest weight of a Serre weight F, we say that F is generic if

$$3 < a - b, b - c < p - 5, \qquad |a - c - p + 2| > 4.$$

Assume that $\overline{r}|_{G_{F_w}}$ is semisimple. Then there is a set of conjectural weights $W^?(\overline{r}|_{G_{F_w}})$ defined in [**Her09, GHS**] which only depends on the restriction of $\overline{r}$ to the inertia subgroups at w.

Theorem 4.4 ([**LLHLMb**]). — *Let $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ be a continuous Galois representation, verifying the Taylor-Wiles conditions. Assume that $\overline{r}|_{G_{F_v}}$ is semisimple and 6-generic at v, that $\overline{r}$ is automorphic of some generic Serre* *weight, and that $\overline{r}$ has split ramification outside p. Then*

$$W_w^{gen}(\overline{r}) = W^?(\overline{r}|_{G_{F_w}}).$$

When $\overline{r}$ is irreducible at each prime above p, this is proven in [**EGH13**] using the technique of weight cycling and without any Taylor-Wiles conditions. The inclusion $W_w^{gen}(\overline{r}) \subset W^?(\overline{r}|_{G_{F_w}})$ (weight elimination) is proven in [**EGH13, HLM17, MP**]. Recent improvements in weight elimination results show that $W_w^{gen}(\overline{r})$ can be replaced by $W_w(\overline{r})$ and 'automorphic of some generic Serre weight' with just 'automorphic' in almost all cases.

There are 9 conjectural weights appearing in $W^?(\overline{r}|_{G_{F_w}})$, 6 of which are called *obvious* weights since they are directly related to the Hodge-Tate weights of "obvious" crystalline lifts of $\overline{r}|_{G_{F_w}}$. The relation between Serre weights of $\overline{r}$ and Hodge-Tate weights of crystalline lifts of $\overline{r}|_{G_{F_w}}$ was first made precise in [**Gee11b**] and the obvious weights are shown to be modular in [**BLGG13**] using global methods (namely, modularity lifting techniques) under the assumption that $\overline{r}$ is modular of a Fontaine-Laffaille weight.

The remaining weights in $W^?(\overline{r}|_{G_{F_w}})$ are more mysterious and are referred to as *shadow weights*. The modularity of the shadow weights lies deeper than that of the obvious weights, in part, because modularity of a shadow weight cannot be detected by modularity lifting alone but requires p-adic information. It is at this point that the computation of the monodromy operator appears to play a critical role. The proof of Theorem 4.4 builds on the Breuil-Mézard philosophy introduced in [**GK14**]. The patching techniques of Gee-Kisin [**GK14**], Emerton-Gee [**EG14**] connect the geometry of the local deformation rings to modularity questions. We use geometric information about the local deformation rings, especially the geometry of their special fibers, to prove the modularity for the shadow weights.

Theorem 4.4 is stated only for $\overline{r}$ which are semisimple above p because those are the only representations for which there is an explicit conjecture. Our computations, together with work of [**HLM17, MP**], suggest a set $W^?(\overline{r}|_{G_{F_w}})$ when $\overline{r}|_{G_{F_w}}$ is not tame, for which the Theorem should hold. This question is addressed in forthcoming [**LLHLMd**]. We also deduce from Table 2 below counterexamples when $\overline{r}|_{G_{F_w}}$ is not semisimple to Conjecture 4.3.2 of [**Gee11b**], which predicts Serre weights in terms of the existence of crystalline lifts. The conjecture is reformulated in [**GHS**, Conjecture 5.1.6].

The local information on the deformation rings (Theorem 4.3), namely the connectedness of their generic fiber, lets us deduce new modularity lifting theorems.

Theorem 4.5 ([**LLHLMb**]). — *Let $r : G_F \to \mathrm{GL}_3(\mathscr{O})$ be a Galois representation and write $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ for its associated residual representation.*

Assume that:

1. *p splits completely in F^+;*
2. *r is conjugate self-dual and unramified almost everywhere;*
3. *for all places $w \in \Sigma_p$, the representation $r|_{G_{F_w}}$ is potentially crystalline, with parallel Hodge type $(2,1,0)$ and with strongly generic tame inertial type $\tau_{\Sigma_p^+} = \otimes_{v\in\Sigma_p^+}\tau_v$;*
4. *$\overline{r}$ verifies the Taylor-Wiles conditions, in particular $\overline{r}$ is absolutely irreducible) and $\overline{r}$ has split ramification;*
5. *$\overline{r} \cong \overline{r}_{p,\imath}(\Pi)$ for a RACSDC representation Π of $\mathrm{GL}_3(\mathbf{A}_F)$ with trivial infinitesimal character and such that $\otimes_{v\in\Sigma_p^+}\sigma(\tau_v)$ is a K-type for $\otimes_{v\in\Sigma_p^+}\Pi_v$.*

Then r is automorphic.

In Theorem 4.5, we do not assume that $\overline{r}|_{G_{F_w}}$ is semisimple nor do we make any potential diagonalizability assumptions. We also allow any tame type, not just principal series types. Assumption (1) can be relaxed to the condition that p is unramified in F^+. This requires new automorphic techniques which will be discussed in §4.3. We believe also that the genericity assumptions on the type τ can be weakened.

4.3. Breuil's Lattice conjecture in dimension three. — In this section we are addressing a problem in local-global compatibility for the *integral* cohomology of arithmetic manifolds, and which can be viewed as a three-dimensional version of Breuil's lattice conjecture ([**Bre14**], Conjecture 1.2, now a theorem by the work of [**EGS15**]). The difficulties which arise are mainly on the representation-theoretic side, and we develop new techniques -a mixture of local and global methods- to classify lattices in tame types.

4.3.1. Breuil's lattice conjecture. — Motivated by Emerton's local-global compatibility for completed cohomology, [**CEG$^+$16**] constructs a candidate for one direction of the p-adic local Langlands correspondence for $\mathrm{GL}_n(L)$. Namely, they associate to any continuous n-dimensional $\overline{\mathbf{Q}}_p$-representation ρ of G_L an admissible Banach space representation $V(\rho)$ of $\mathrm{GL}_n(L)$ by patching completed cohomology. However, the construction depends on a choice of global setup, and one expects it to be a deep and difficult problem to show that the correspondence $\rho \mapsto V(\rho)$ is purely local.

In [**Bre14**], Breuil formulates a conjecture on lattices in tame types cut out by completed cohomology of Shimura curves which is closely related to the local nature of $V(\rho)$. This conjecture was proven subsequently in the groundbreaking work of Emerton–Gee–Savitt [**EGS15**]. The first main theorem of this section is a generalization of Breuil's conjecture to three-dimenional Galois representations and the completed cohomology of $U(3)$-arithmetic manifolds.

We keep the global input of §3. In particular F/F^+ is a CM extension which *we moreover assume to be everywhere unramified* and we have fixed a prime

TABLE 1. **Shapes of Kisin modules over F**

Shape	$\mathrm{Mat}(\varphi_{\overline{\mathfrak{M}}})$	Shape	$\mathrm{Mat}(\varphi_{\overline{\mathfrak{M}}})$
$\alpha\beta\alpha\gamma$	$\begin{pmatrix} v^2\bar{c}_{11}^* & 0 & 0 \\ v^2\bar{c}_{21} & v\bar{c}_{22}^* & 0 \\ \bar{c}_{31}v+\bar{c}'_{31}v^2 & v\bar{c}_{32} & \bar{c}_{33}^* \end{pmatrix}$	$\beta\gamma\alpha\gamma$	$\begin{pmatrix} v\bar{c}_{11}^* & v\bar{c}_{12} & 0 \\ 0 & v^2\bar{c}_{22}^* & 0 \\ v\bar{c}_{31} & \bar{c}_{32}v+\bar{c}'_{32}v^2 & \bar{c}_{33}^* \end{pmatrix}$
$\beta\alpha\gamma$	$\begin{pmatrix} 0 & v\bar{c}_{12}^* & 0 \\ v^2\bar{c}_{21}^* & 0 & 0 \\ v\bar{c}_{31}+v^2\bar{c}'_{31} & v\bar{c}_{32} & \bar{c}_{33}^* \end{pmatrix}$	$\alpha\beta\gamma$	$\begin{pmatrix} v^2\bar{c}_{11}^* & 0 & 0 \\ v\bar{c}_{21}+v^2\bar{c}'_{21} & 0 & \bar{c}_{23}^* \\ v^2\bar{c}_{31} & v\bar{c}_{32}^* & 0 \end{pmatrix}$
$\alpha\beta\alpha$	$\begin{pmatrix} 0 & 0 & v\bar{c}_{13}^* \\ 0 & v\bar{c}_{22}^* & v\bar{c}_{23} \\ v\bar{c}_{31}^* & v\bar{c}_{32} & v\bar{c}_{33} \end{pmatrix}$		
$\alpha\beta$	$\begin{pmatrix} 0 & 0 & v\bar{c}_{13}^* \\ v\bar{c}_{21}^* & 0 & v\bar{c}_{23} \\ 0 & v\bar{c}_{32}^* & v\bar{c}_{33} \end{pmatrix}$	$\beta\alpha$	$\begin{pmatrix} 0 & v\bar{c}_{12}^* & 0 \\ 0 & 0 & v\bar{c}_{23}^* \\ v\bar{c}_{31}^* & v\bar{c}_{32} & v\bar{c}_{33} \end{pmatrix}$
α	$\begin{pmatrix} 0 & v\bar{c}_{12}^* & 0 \\ v\bar{c}_{12}^* & v\bar{c}_{22} & 0 \\ 0 & 0 & v\bar{c}_{33}^* \end{pmatrix}$	id	$\begin{pmatrix} v\bar{c}_{11}^* & 0 & 0 \\ 0 & v\bar{c}_{22}^* & 0 \\ 0 & 0 & v\bar{c}_{33}^* \end{pmatrix}$

In this table we classify mod p Kisin modules $\overline{\mathfrak{M}}$ with tame generic descent data according to their *shapes*. In particular, the second and fourth column describe the matrix of the Frobenius on $\overline{\mathfrak{M}}$ for a convenient choice of an eigenbasis on $\overline{\mathfrak{M}}$. Recall that α, β, γ are the standard generators of the affine Weyl group of SL_3, where α and β correspond to permutations (12), (23) and γ corresponds to a translation by the longest root followed by a permutation by (13). Moreover we have $\bar{c}_{ik}, \bar{c}'_{ik} \in \overline{\mathbf{F}}$ and $\bar{c}_{ik}^* \in \overline{\mathbf{F}}^\times$.

TABLE 2. **Deformation rings with monodromy**

Shape	Condition on $\overline{\mathfrak{M}}$	$R_{\overline{\rho}}^{(2,1,0),\tau}$
$\alpha\beta\alpha\gamma$	$\bar{c}_{31} = 0$	$\mathscr{O}[\![x_{11}^*, x_{22}^*, x_{33}^*, x_{21}, x_{31}', x_{32}]\!]$
$\beta\gamma\alpha\gamma$	$(e-b+c)\bar{c}_{32}\bar{c}_{11}^* = (e-a+c)\bar{c}_{12}\bar{c}_{31}$	$\mathscr{O}[\![x_{11}^*, x_{22}^*, x_{33}^*, x_{12}, x_{31}, x_{32}']\!]$
$\beta\alpha\gamma$	$\bar{c}_{31} = 0$	$\mathscr{O}[\![y_{11}, y_{22}, x_{12}^*, x_{21}^*, x_{33}^*, x_{31}', x_{32}]\!]/(y_{11}y_{22} - p)$
$\alpha\beta\gamma$	$\bar{c}_{21} = 0$	$\mathscr{O}[\![y_{22}, y_{33}, x_{11}^*, x_{21}', x_{23}^*, x_{31}', x_{32}^*]\!]/(y_{22}y_{33} - p)$
$\alpha\beta\alpha$	$(a-b)\bar{c}_{23}\bar{c}_{32} - (a-c)\bar{c}_{22}^*\bar{c}_{33}' \neq 0$	$\mathscr{O}[\![x_{32}, x_{23}, x_{33}', x_{31}^*, x_{22}^*, x_{13}^*]\!]$
$\alpha\beta\alpha$	$(a-b)\bar{c}_{23}\bar{c}_{32} - (a-c)\bar{c}_{22}^*\bar{c}_{33}' = 0$	$\mathscr{O}[\![x_{11}, x_{32}, x_{23}, y_{33}', x_{31}^*, x_{22}^*, x_{13}^*]\!]/(x_{11}y_{33}' - p)$
$\alpha\beta$	$\bar{c}_{33}' \neq 0$	$\mathscr{O}[\![y_{31}, x_{22}, x_{23}', x_{33}', x_{21}^*, x_{13}^*, x_{32}^*]\!]/(y_{31}x_{22} - p)$
$\alpha\beta$	$\bar{c}_{33}' = 0$	$\mathscr{O}[\![y_{31}, x_{22}, x_{12}, x_{23}', y_{33}', x_{21}^*, x_{13}^*, x_{32}^*]\!]/(y_{31}x_{22} - p, x_{12}y_{33}' - p)$
$\beta\alpha$	$\bar{c}_{32} \neq 0$	$\mathscr{O}[\![x_{22}', y_{13}, x_{32}, x_{33}', x_{31}^*, x_{12}^*, x_{23}^*]\!]/(x_{22}'y_{13} - p)$
$\beta\alpha$	$\bar{c}_{32} = 0$	$\mathscr{O}[\![x_{11}, x_{22}', y_{32}, y_{13}, x_{33}', x_{31}^*, x_{12}^*, x_{23}^*]\!]/(x_{22}'y_{13} - p, x_{11}y_{32} - p)$

For each shape of length ≥ 2 we describe the potentially crystalline deformation ring with tame, generic descent data. For each shape, the coefficients $\bar{c}_{ij}$ in the second column are those appearing in the corresponding row in Table 1. In the third column we put a formally smooth modification of the potenially crystalline deformation ring of $\overline{\rho}$, with tame type τ and Hodge-Tate weights $(2,1,0)$.

$v|p$ of F^+ which splits completely $v = ww^c$ in F. Let $r : G_F \to \mathrm{GL}_3(\overline{\mathbf{Q}}_p)$ be a continuous Galois representation and λ be the Hecke eigensystem corresponding to r, which appears in the cohomology of a $U(3)$-arithmetic manifold. Let $\widetilde{S}(U^v, \mathscr{O})$ be the integral v-adically completed cohomology. One expects completed cohomology to realize a global p-adic Langlands correspondence generalizing the case of $\mathrm{GL}_2(\mathbf{Q})$, that is, $\widetilde{S}(U^v, \mathscr{O})[\lambda]$ corresponds to $r|_{G_{F_w}}$ via a hypothetical p-adic local Langlands correspondence. In particular, the globally constructed object $\widetilde{S}(U^v, \mathscr{O})[\lambda]$ should depend only on $r|_{G_{F_w}}$.

Suppose that r is minimally ramified away from p and tamely potentially crystalline with Hodge–Tate weights $(0,1,2)$ at each place above p. Suppose that τ corresponds to the Weil–Deligne representations associated to $r|_{G_{F_w}}$ under the inertial local Langlands correspondence. If r is modular, then by classical local-global compatibility, τ appears as a $\mathrm{GL}_3(\mathscr{O}_{F_w})$ subrepresentation of $\widetilde{S}(U^v, \mathscr{O})[\lambda][1/p]$, with multiplicity one.

Theorem 4.6 (**[LLHLMc]**). — *Assume that p is unramified in F^+, and that $\overline{r}$ satisfies Taylor–Wiles hypotheses, has split ramification, and is semisimple and sufficiently generic at places above p. Then, the lattice*

$$\tau^0 := \tau \cap \widetilde{S}(U^v, \mathscr{O})[\lambda] \subset \tau$$

depends only on $r|_{G_{F_w}}$.

We also prove the following "mod p multiplicity one" result:

Theorem 4.7 (**[LLHLMc]**). — *With hypotheses as in Theorem 4.6, let τ^σ be a lattice in τ such that the reduction $\overline{\tau}$ has a irreducible upper alcove cosocle, then $\mathrm{Hom}_{K_v}(\overline{\tau}, S(U^v, \mathbf{F})[\lambda])$ is one dimensional.*

These theorems should be compared to Theorem 8.2.1 and Theorem 10.2.1 in [**EGS15**] in dimension two. (In the special case where p is split in F^+ and $\overline{r}$ is irreducible above p, these were first proven by Daniel Le in [**Le**]).

The main ingredients used in [**EGS15**] are the Taylor–Wiles patching method, the geometric Breuil–Mézard conjecture for potentially Barsotti–Tate Galois deformation rings (building on work of [**Bre14**]), and a classification of lattices in tame types (extending [**Bre14, BP12**]). Before the work [**LLHLMc**], only the first of these tools was available in the case of GL_3. The analogue of potentially Barsotti–Tate Galois deformation is potentally crystalline deformation rings with Hodge–Tate weights $(2,1,0)$. In [**LLHLMb**], we develop a technique for computing these Galois deformation rings with tame descent. We discuss in §4.3.2 the geometric Breuil–Mézard conjecture for these rings. The representation theoretic results are discussed in §4.3.3.

It is worth mentioning several key differences which distinguish our situation from [**EGS15**]. Breuil's conjecture for GL_2 gave an explicit description of the lattice τ^0 in terms of the Dieudonné module of $r|_{G_{F_w}}$. We prove abstractly that τ^0 is "purely local" but without giving any explicit description of the lattice. The lattice τ^0 is determined by the parameters of the Galois deformation ring but in a complicated way.

Let τ^σ be a lattice in τ whose reduction has irreducible cosocle σ. To prove Theorem 4.7, we show that a certain (minimal) patched module $M_\infty(\tau^\sigma)$ is free of rank one over the local Galois deformation ring (with patching variables) $R_\infty(\tau)$. In fact, this result is also a key step in our proof of Theorem 4.6. In [**EGS15**], the analogue of this result is Theorem 10.1.1 where they show that the patched module of any lattice with irreducible cosocle is free of rank 1. In our situation, it is no longer true that all such patched modules are cyclic. Rather, this is only true when the cosocle σ is upper alcove in every embedding. As a consequence of this, one can deduce that the isomorphism class as an $R_\infty(\tau)$-module of $M_\infty(\tau^\sigma)$ is purely local for any lattice τ^σ, however, it need not be free.

For the proof that $M_\infty(\tau^\sigma)$ is free of rank one when σ is upper alcove, we induct on the complexity of the deformation ring. The simplest deformation rings resemble those for GL_2 and so we follow the strategy similar to [**EGS15**]. For the most complicated deformation rings, we build up $M_\infty(\tau^\sigma)$ from its subquotients relying on the description of the submodule structure of reduction $\overline{\tau}^\sigma$ discussed in 4.3.3 and crucially intersection theory results for components of mod p fiber of the Galois deformation ring.

4.3.2. Serre weight and Breuil–Mézard conjectures. — We now get back to the global setting in 4.2.2, *with the weaker assumption that p is unramified in* F^+, and all places $v|p$ of F^+ split completely in F. In what follows we fix a finite place $v|p$ of F^+ as well as a finite place $w|v$ of F.

We fix a mod p Galois representation $\overline{r} : G_F \to \mathrm{GL}_3(\mathbf{F})$ whose corresponding Hecke eigensystem $\mathfrak{m}_{\overline{r}}$ appears in the mod p cohomology of a $U(3)$-arithmetic manifold. One expects $S(U^v, \mathbf{F})[\mathfrak{m}_{\overline{r}}]$ to correspond to $\overline{r}|_{G_{F_w}}$ via a hypothetical mod p local Langlands correspondence. Furthermore, if we let $W_w(\overline{r})$ be the set of irreducible $\mathrm{GL}_3(\mathbf{Z}_p)$-representations whose duals occur in the $\mathrm{GL}_3(\mathbf{Z}_p)$-socle of $S(U^v, \mathbf{F})_{\mathfrak{m}_{\overline{r}}}$ and $\overline{r}|_{G_{F_w}}$ is tamely ramified, [**Her09, GHS**] predict that $W_w(\overline{r}) = W^?(\overline{r}|_{G_{F_w}})$, for some set $W^?(\overline{r}|_{G_{F_w}})$ explicitly defined in terms of $\overline{r}|_{I_{F_w}}$.

We have the following version of the weight part of Serre's conjecture:

Theorem 4.8 **([LLHLMc])**. — *Assume that p is unramified in F^+, and that $\overline{r}$ satisfies Taylor–Wiles hypotheses, has split ramification, and is semisimple and sufficiently generic at places above p. Then $W_w(\overline{r}) = W^?(\overline{r}|_{G_{F_w}})$.*

In [**LLHLMb**], we proved this theorem with the additional assumption that p is split in F^+. The strategy is to show the numerical Breuil–Mézard conjecture for the simplest deformation rings (shape has length at least two) using [**LLHLMb**, §6.2]. The key new tool is a more conceptual and robust combinatorial technique for computing the intersection between the predicted weights $W^?(\overline{r}|_{G_{F_w}})$ and the Jordan–Hölder factors of a type. This allows us to inductively prove that all predicted weights are modular.

Using a patching functor which is constructed globally, we show that the generic fibers of tamely potentially crystalline deformation rings of Hodge–Tate weight $(0, 1, 2)$ are connected for $\overline{\rho}$ generic and deduce the full numerical Breuil–Mézard conjecture for these Galois deformation rings (all multiplicities for modular Serre weights are one). Using the numerical formulation, we prove the following geometric version of the Breuil–Mézard conjecture (cf. [**EG14**]).

Theorem 4.9 **([LLHLMc])**. — *There is a unique assignment $\sigma \mapsto \mathfrak{p}(\sigma)$ taking Serre weights $\sigma \in W^?(\overline{r}|_{G_{F_w}})$ to prime ideals in the unrestricted deformation ring $R^{\square}_{\overline{r}|_{G_{F_w}}}$ such that the special fiber* Spec $(\overline{R}^{\tau,\square}_{\overline{r}|_{G_{F_w}}})$ *of a generic tamely potentially crystalline deformation ring of Hodge–Tate weight $(0, 1, 2)$ is the reduced underlying subscheme of*

$$\bigcup_{\sigma \in W^?(\overline{r}|_{G_{F_w}}) \cap \mathrm{JH}(\overline{\tau})} \mathrm{Spec}\ (R^{\square}_{\overline{r}|_{G_{F_w}}}/\mathfrak{p}(\sigma)).$$

Moreover, this is compatible with any patching functor.

4.3.3. Representation theory results. — In order to deduce Breuil's lattice conjecture from the Breuil-Mézard conjecture we need and prove new results on integral structures in Deligne-Lusztig representations, which may be of independent interest. The main result is a classification of integral lattices with irreducible cosocle in tame types, by means of an *extension graph*, which plays a key role in the proofs of our results on modular representations theory.

We now briefly describe the extension graph. In [**LLHLMc**, §2], we introduce a graph on the set of regular Serre weights (with fixed central character), with vertexes corresponding to regular Serre weights and adjacency between vertexes described in a combinatorially explicit way. We then show that two vertexes are adjacent if and only if the corresponding Serre weights have a nontrivial $\mathrm{GL}_3(\mathbf{F}_q)$-extension between them, justifying the terminology. This gives a natural notion of *graph distance* $\mathrm{d}_{\mathrm{gph}}$ between two regular Serre weights.

Theorem 4.10 **([LLHLMc])**. — *The Jordan–Hölder factors of $\overline{\tau}$ occur with multiplicity one. Suppose that $\sigma, \sigma' \in \mathrm{JH}(\overline{\tau})$ and that $\mathrm{d}_{\mathrm{gph}}(\sigma, \sigma') = d$. Then*

1. *σ' is a direct summand of the d-th layer of the cosocle filtration of $\overline{\tau}^{\sigma}$;*
2. *every extension that might occur between layers of the cosocle filtration of $\overline{\tau}^{\sigma}$ described in (1) does occur; and*
3. *if $\tau^{\sigma'} \subseteq \tau^{\sigma}$ is a saturated inclusion of lattices, then $p^d\tau^{\sigma} \subseteq \tau^{\sigma'}$ is also a saturated inclusion of lattices.*

The argument is involved, using a mixture of local and global techniques, but we can distinguish two main steps in its proof. In the first step we prove the first two items of Theorem 4.10 in the case when σ is a lower alcove weight of defect zero. The proof uses methods from the modular representation theory of algebraic groups, embedding $\overline{\tau}^{\sigma}$ in a Weyl module with non-p-restricted highest weights. The key local argument is a careful study of the restriction of algebraic representations to rational points, letting us constrain the submodule structure of the relevant part of a Weyl module restricted to $\mathrm{GL}_3(\mathbf{F}_q)$ in terms of the extension graph. This method does not work for all weights $\sigma \in \mathrm{JH}(\overline{\tau})$, as the corresponding lattices will not always have simple socle, and thus can not be embedded into a Weyl module.

In the second step, we reduce the theorem for the remaining lattices to the case treated in the first step. We relate the first two items and the last item of Theorem 4.10. The last item, a statement in characteristic zero, is amenable to an inductive analysis. First, we show that for a fixed weight $\sigma \in \mathrm{JH}(\overline{\tau})$, item (3) of Theorem 4.10 actually implies the other two items. This crucially uses Theorem 4.10(3) in the case $d = 1$, which is proved using the computation of deformation rings in [**LLHLMb**], combinatorics of modular weights, and the Kisin–Taylor–Wiles patching method. This argument follows the suggestion in [**EGS15**], §B.2 that tamely potentially crystalline deformation rings strongly reflect aspects of local representation theory through global patching constructions.

Next, we show that the first two items of Theorem 4.10 applied to τ and its dual in the case of lower alcove defect zero weights imply Theorem 4.10(3) in the case of lower alcove defect zero weights. From this as a starting point, an inductive argument proves Theorem 4.10(3), thus concluding the proof of Theorem 4.10.

5. Some perspectives

5.1. The mod-p local correspondence for $\mathbf{GL}_2(L)$. — In future research we plan to investigate the local correspondence for Hilbert modular forms. This has already been undertaken in collaboration with B. Schraen.

As a first step, we study "natural" procedures to associate (φ, Γ)-modules to smooth $\mathbf{GL}_2(L)$-representations, for an unramified extension $L/\mathbf{Q}_p$. In ongoing work ([**MS**]) we compare Breuil's construction [**Bre11**] with the current conjectures on the mod-p local Langlands for *supersingular* representations.

Proposed Theorem 5.1 ([**MS**]). — *Let $\overline{\rho} : G_L \to \mathbf{GL}_2(\mathbf{F})$ be a semisimple Galois representation. Let $\pi(\overline{\rho})$ be any of the smooth representations associated to $\overline{\rho}$ by Breuil and Paskunas (*[**BP12**]*, Theorem 1.5).*

If $f \leq 3$ and $\overline{\rho}$ is suitably generic, then

$$\mathrm{ind}_L^{\otimes \mathbf{Q}_p}(\overline{\rho}) \hookrightarrow V(\pi(\overline{\rho}))$$

where V denotes the generalized Montréal functor of [**Bre15a**] *and $\mathrm{ind}_L^{\otimes \mathbf{Q}_p}(\overline{\rho})$ is the tensor induction of $\overline{\rho}$.*

Our procedure crucially relies on a deep understanding of certain finite length $\mathbf{GL}_2(\mathscr{O}_L)$-representations (the natural generalization of those provided by [**Mor11**]), local/global compatibility conjectures and the properties of functors on F-torsion modules over $\mathbf{F}[\![\mathbf{Z}_p]\!]$.

We aim at giving a more conceptual insight to the computations in [**MS**], with special emphasis to those representations appearing in the cohomology of algebraic automorphic forms. In particular, by suitably generalizing the analysis of the weight spaces of irreducible $\mathrm{GL}_2(\mathbf{F}_q)$-representation and using the necessary genericity assumption to have multiplicity one, we hope to obtain

Proposed Theorem 5.2 ([**MS**]). — *Let $\overline{\rho} : G_L \to \mathbf{GL}_2(\mathbf{F})$ be a semisimple Galois representation. Let $\pi(\overline{\rho})$ any of the smooth representations associated to $\overline{\rho}$ by Breuil and Paskunas (*[**BP12**]*, Theorem 1.5). Then $V(\pi(\overline{\rho}))$ contains a direct summand of $\mathrm{ind}_L^{\otimes \mathbf{Q}_p}(\overline{\rho})$. In particular, $V(\pi(\overline{\rho})) \neq 0$.*

Moreover, if the inertial weights r_i of $\overline{\rho}$ verify $\frac{f}{2} < r_i < p - \frac{f}{2}$, then

$$\mathrm{ind}_L^{\otimes \mathbf{Q}_p}(\overline{\rho}) \hookrightarrow V(\pi(\overline{\rho}))$$

The problem of whether the injection is an isomorphism is extremely subtle. It relies on the torsion properties of the torsion F-module which is naturally attached to a smooth $\mathbf{GL}_2(L)$-representation. Under suitable bounds on the torsion submodule we strongly believe the isomorphism holds true, but due to the huge number of parameters appearing in supersingular representations it is currently unclear how to impose the F-torsion requirement in purely representation theoretical terms. The language of derived geometry, especially having an explicit realization of these parameters in terms of homological dimension of supersingular reprsentation, would be of particular relevance.

5.2. Local-global compatibility. — The non-semisimple cases in Serre-type conjectures are of substantial interest in the emerging p-adic and mod p Langlands program: in the case of GL_2 this has been initiated by Breuil, exhibiting the interesting combinatorial and representation theoretic structure which are the foundation of [**BP12**] on mod p Langlands for $\mathrm{GL}_2(L)$.

Remark 5.3 ([**LLHLMd**]). — *In the ongoing work* [**LLHLMd**] *we are formulating conjectural sets of weights* $W^?(\overline{\rho})$ *for non-semisimple* $\overline{\rho} : G_L \to \mathrm{GL}_3(\mathbf{F})$*, investigate the structure of* $W^?(\overline{\rho}) \subset W^?(\overline{\rho}^{\mathrm{ss}})$*, explore patterns for* GL_3*, trying to highlight the behavior of a mod p Langlands correspondence in the unramified generic setting.*

Note that in the non-semisimple case the set of modular weights is not described in terms of mod p reduction of automorphic types. We expect a geometric description in terms of *shapes* and closure relations in the Bruhat stratification of a local model. To this aim we define a notion of *admissible shapes*, and prove the semicontinuity of shapes (hence the realization of geometric weights in terms of closure relations). An explicit computation on Schubert cells shows that the admissible shapes specialize to any other admissible shape with strictly lower length.

Another research direction is focused in developing a "monodromy local model" M^{∇} which is obtained as the p-saturation of the equations issued from the *algebraic* monodromy conditions on the local model M introduced by Kisin.

By results of Elkik ([**Elk73**]) the specializations of M^{∇} surject onto the Galois deformation spaces. Moreover, M^{∇} is topologically flat and its generic fiber is isomorphic to a flag variety ([**LLHLMa**]). The key problem is to prove it is "unibranch", and for that we suggest a strategy which is reminescent of the proof of the coherence conjecture of Pappas and Rapoport [**Zhu14**].

Aiming at a deeper understanding of the mod p local Langlads correspondence, we plan to develop the results of [**LMS**] to both the ramified and higher higher rank.

In the rank 2 case the goal would be a better understanding of the behavior of cohomological supersingular representation at a deeper congruence level. To this aim, we plan to give a detailed study of the submodule structure of projective envelopes in terms of the extension graph introduced in [**LLHLMb**]. It turns out that the extension graph allows one to describe the structure of wildly ramified, principal series types, in a similar way as it did already in the case of tame types. We expect this analogy to hold true also for cuspidal types, hence leading to a struture theorem for ramified projective envelopes. We would hope to deduce a freeness result for the patched module of a $K/K(n)$-projective envelope of a weight, following the techniques of [**LMS**].

In higher dimension we also expect a full characterization of the $K(1)$-invariants for $U(3)$-arithmetic manifolds. We expect the multiplicities of weights to depend explicitly on the p-restricted alcove the weight belongs to. More importantly, we expect the $K(1)$-invariants to depend only on the local Galois parameter at p. In the following proposed theorem we keep the general global setting recalled at the beginning of 3.

Proposed Theorem 5.4 **([LLHM])**. — *Keep the assumptions of Theorem 4.4. Assume for simplicity that p splits completely in F. Let σ be a modular weight for $\overline{r}$ and write P_σ for the $\mathbf{GL}_3(\mathbf{F}_p)$-projective envelope of σ.*

Then the $\mathbf{GL}_3(\mathbf{F}_p)$-representation $S(K^vK(1),\mathbf{F})[\mathfrak{m}_{\overline{r}}]$ depends only on $\overline{r}|_{G_{F_w}}$ and we have

$$\dim \mathrm{Hom}_{\mathbf{F}}(P_\sigma, S(K^vK(1),\mathbf{F})[\mathfrak{m}_{\overline{r}}]) = \begin{cases} 3 & \text{if } \sigma \text{ is lower alcove} \\ 1 & \text{if } \sigma \text{ is upper alcove} \end{cases}$$

The proof of the proposed Theorem follows from computations in local rings attached to sufficiently big subvarieties in the affine flag variety of $\mathrm{GL}_{3/\mathbf{F}_q}$. We remark that these local rings do not appear as deformation rings attached to classical automorphic types, but they rather surject onto the usual potentyally cystalline deformation. In particular these rings contain *all* modular weights and we are able to identify which weight correspond to which ideal in the support of any patched module.

Finally, we aim at developping a mod p Langlands functoriality in the context of unitary groups. Note that for unramified groups a notion of modular Serre weights has been introduced in [**GHS**]; for semisimple Galois parameters the set of modular weights can again be recovered in terms of automorphic types. In ongoing work with K. Koziol we introduce the notion of *base change* for Serre weight. This notion is compatible with the classical automorphic base change for unitary groups; in particular in the rank one case (building on the classical work of Rogawski [**Rog90**]) this compatibility let us deduce the weight part of Serre conjecture for unramified $\mathbf{U}(1,1)$ from the analogous result of Gee and collaborators on Hilbert modular forms (cf. [**GLS14**]):

Proposed Theorem 5.5 **([KM])**. — *Let F/F^+ be a CM field where p is unramified and let $\overline{r}: G_{F^+} \to {}^C\mathbf{U}_2(\mathbf{F})$ be a continuous, absolutely irreducible Galois representation such that $d\circ\overline{r} = \omega$ (here, ${}^C\mathbf{U}_2(\mathbf{F})$ denotes the cohomological dual group for the unitary group in two variable $\mathbf{U}_2$). Assume $\overline{r}$ satisfies the standard Taylor-Wiles conditions and the local parameter at $v|p$ is 2-generic. Then $W_v(\overline{r})$ depends only on $\overline{r}|_{I_{\mathbf{Q}_p}}$ as predicted by [**GHS**] and moreover*

$$F \in W_p(\overline{r}) \quad\Longleftrightarrow\quad \mathrm{BC}(F) \in W_v(\overline{r}|_{G_F}).$$

For higher rank we are currently studying the behavior of K-types with respect to automorphic base change, and giving an alternative description of base change in terms of affine Weyl group elements, which is a more suitable on the Galois representation side (cf. [**LLHL**]). We expect the procedure of [**KM**] to be completely general, and letting us establish a mod p-Langlands functoriality for Serre weights.

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